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# MATHEMATICS FOR CANADIANS

(*National Film Board*

***The Library of Parliament at Ottawa provides an interesting example of the use of geometric forms in architecture.***

# MATHEMATICS FOR CANADIANS

## Book II

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***Mathematics plays an important role in designing, constructing, and operating many modern marvels of science.***

Top: T.C.A. "North Star" over Ottawa.
*(Courtesy Trans-Canada Air Lines)*

Centre: Thousand Island Bridge from Georgian Island.
*(National Film Board Photo)*

Bottom: Modern C.N.R. locomotive.
*(Courtesy C.N.R.)*

CHAPTER I

# RE-UNION WITH ALGEBRA

*"Mathematics has a light and wisdom of its own, . . . and it will richly reward any intelligent human being to catch a glimpse of what mathematics means . . ."*[1]

—E. T. Bell.

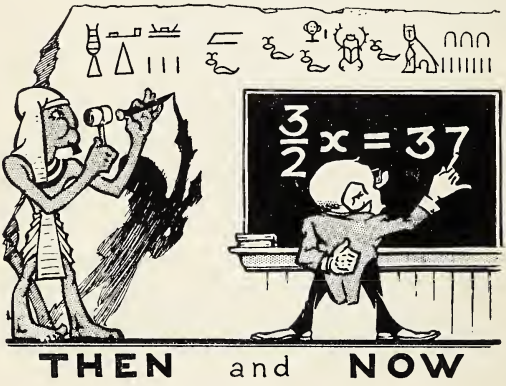
## *A Few Brief Reminders*

This book assumes that you have already made the step from arithmetic to algebra. In a few brief months, you spanned with giant strides the work of centuries. This does not mean that modern students are supermen compared with the plodders of yesterday. It merely serves to emphasize the difference between the difficulties of discovery and those of learning known facts.

The race did not develop algebra in the orderly way in which students meet it today. Through the centuries many lonely rivers of information made their slow and often halting way to empty finally into the sea of algebra. The Hindus, Arabs, Egyptians, Greeks and Europeans all made important contributions to the growth of mathematics. We take our numbers for granted nowadays, but their development is the result of the combined efforts of many great minds over thousands of years. Ideas were discovered, lost, re-discovered, extended, until the modern days of algebraic notation were reached. While many men of different races experimented with various notations, it is usually agreed that some of the most outstanding work was that of Vieta, a French lawyer who lived from 1540 to 1630. It was Vieta who first used the letters of the alphabet to stand for general numbers.

[1] From *Development of Mathematics*, by permission of the publishers, McGraw Hill Book Company Inc., New York.

The results of all this creative thinking, and of much mor that dates from the nineteenth and twentieth centuries, hav greatly refined and simplified the methods of algebra. Problem that used to baffle geniuses can now be attacked successfull by people with more modest gifts. You can probably reca some of the ideas and methods of your first course in algebr It will be well for you to pause and think about some of th ways in which algebra gives added power to arithmetic.




## *Advantages of Algebra*

You stepped from arithmetic to algebra when you firs used letters to stand for numbers. These new letter-numbers once you learned how to deal with them, opened a way to bette methods of attacking some everyday problems. Here ar four great advantages of algebra which are readily illustrate from your own experience.

**1. Algebra is concise.** The letter-numbers provide wonderful new shorthand which is clear, compact, and ver convenient. Perhaps you remember the following compari son:

| *English Statement* | *Algebraic Statement* |
| --- | --- |
| *The number of units in the area of a circle is equal to pi times the square of the number of units in its radius.* | $A = \pi r^2$ |

**2. Algebra is ideal for expressing relations.** Life teems with examples of quantities which are related; the time required for a journey is related to the average speed, the amount of gasoline used is related to the speed of the car, the volume of a tin can is related to the area of its base and the height, and so on. The formulas of algebra give us many of these relationships in a very handy form.

**3. Algebra helps us in solving problems.** From your experience with the equations of algebra, you know that an algebraic equation often provides a way of finding an unknown number. The great wealth of problems involving unknown numbers gives wide scope for the use of algebra.

**4. Algebra can picture dependence.** The formula expresses many relations in convenient form; the graph depicts such relations so as to show vividly how one quantity depends on another. Virtually all newspapers and magazines use graphs. Their interpretation should be one of the skills of the well-informed citizen.

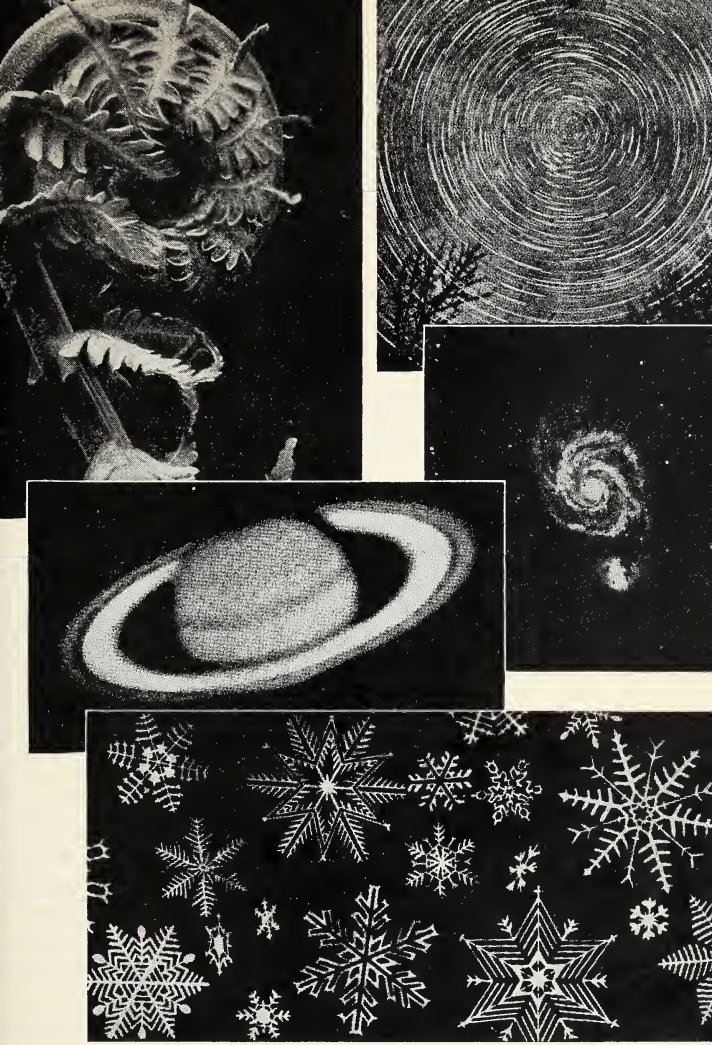
## *Usefulness of Algebra to the Citizen*

In the introduction to Book I, we discussed the question *Why should I study mathematics?* One of the reasons give in answer to this question was the usefulness of mathematics Let us briefly review some of the argument in support of thi reason.

If you plan to be a scientist, engineer, architect, actuary economist, statistician, aviator, navigator, etc., then yo must know that mathematics is a necessity for your futur The formula, the equation, and the graph are basic tools such professions. Bridges, railroads, aircraft, skyscraper dams and such structures are not designed by men ignoran of mathematics. But what of those of you who plan entirel different ways of making your living? To you also, mathe matics has something of great value to offer.

Ours is the Age of Science; mathematics is the basis f many of the sciences. Surely it must follow that some know ledge of mathematics helps the citizen appreciate more full the world about him. Some of the facts of algebra are usef in your other school subjects and in everyday life as wel The formulas of physics, chemistry, and biology; the buyin of canned fruit, eggs, and oranges; the problems of buying the installment plan; questions arising in connection wit insurance and taxes;—all of these offer opportunities for th use of elementary algebra and arithmetic. A recent listing b the U.S. Army of about 600 jobs, many of which have civilia counterparts, discloses a rather imposing array of minimu mathematical needs.

To these practical uses must be added opportunities th may prove of much greater value. An understanding of th ways in which certain quantities depend on other quantities, constant challenge to think clearly and make correct judgment and a contact with the beauty of order and form are some the other worth-while results of mathematical studies.



***Here are some examples of geometric forms found in nature.***

Top left: spiral fern. Top right: circular trails of circumpolar stars (Ritchey) *(From "Introduction to Astronomy" by F. R. Moulton—Macmillan).* Centre left: the planet Saturn (Mount Wilson Observatory) *(From "Through Space and Time" by Sir Jas. H. Jeans—Macmillan).* Centre right: spiral nebula (Mount Wilson Observatory) *(From "Through Space and Time" by Sir Jas. H. Jeans—Macmillan).* Bottom: snow crystals.

Of course, you may be one of the many who enjoy mathematics for its own sake, finding fun and satisfaction in solving its problems, practical or otherwise.

### *What Lies Ahead?*

Naturally, we want to push on to new things. You have had some practice already with a number of the basic skills and principles of algebra. These might be listed as follows:

**YOU SHOULD KNOW**

(1) *How to substitute.*
(2) *How to add, subtract, multiply, and divide signed numbers, both numerical and literal.*
(3) *How to use parentheses.*
(4) *How to use formulas.*
(5) *How to solve some equations.*
(6) *How to solve certain types of problems.*

Have you forgotten some of these things? Don't worry. Ideas and skills, like other tools, have a way of getting rusty when they are out of use for some time. However, if we are to advance our knowledge of algebra, the need for these abilities will appear almost at once. One aim of this book is to extend your knowledge of algebra and arithmetic; the other is to introduce you to some new and interesting methods in geometry. In order to realize the first of these aims, we shall now give you a chance to recall some of the algebra of your first course.

You will find immediately following, sets of exercises designed to help you review the six basic skills listed above. You may not need to use all of the sets provided. As soon as you feel confident of your ability, move on to the new work.

## SOME RULES WORTH REVIEWING

**1. *Rule for adding signed numbers.***

*(a) If the numbers have* ***like*** *signs, add their numerical values and attach the common sign.*

*(b) If the numbers have* ***unlike*** *signs, find the difference of the numerical values, and attach the sign belonging to the number with the greater numerical value.*

**2. *Rule for subtracting signed numbers.***

*Mentally change the sign of the number being subtracted and add the resulting number.*

**3. *Rule for multiplying signed numbers.***

*If the numbers have* ***like*** *signs, their product is positive. If the numbers have* ***unlike*** *signs, their product is negative. In either case, the numerical value of the product is found by multiplying the numerical values of the numbers.*

**4. *Rule for dividing signed numbers.***

*If the numbers have* ***like*** *signs, their quotient is positive. If the numbers have* ***unlike*** *signs, their quotient is negative. In either case, the numerical value of the quotient is found by dividing the numerical values of the numbers.*

**5. *Rule for adding like terms.***

*The sum of like terms is another like term whose coefficient is found by adding those of the given terms.*

**6. *Rule for subtracting like terms.***

*The difference of like terms is another like term whose coefficient is found by subtracting those of the given terms.*

**7. *Rule for multiplying monomials.***

*The product of two monomials equals the product of their numerical factors times the product of their literal factors.*

### 8. *Rule for dealing with fractions.*

*The terms of a fraction may be divided or multiplied by the same number (not 0) without changing the value of the fraction.*

## REVIEW SET No. 1

If $x=2$, $y=1$, and $z=-3$, evaluate the following:

**1.** $x^2+y^2$.

**2.** $x^2-z^2$.

**3.** $x^3-z^3$.

**4.** $3x^2-yz$.

**5.** $2x^2-5x+3$.

**6.** $z^2-4z+3$.

**7.** $\dfrac{x+y}{x-y}$.

**8.** $\dfrac{x^2+y^2+z^2}{x+y-z}$.

**9.** $x^3+y^3+z^3-3xyz$

Add:

**10.** $-11$, $6$

**11.** $-7.3$, $-2.1$

**12.** $-\frac{3}{4}$, $\frac{1}{2}$

**13.** $78$, $-65$

**14.** $8ab$, $3ab$

**15.** $7m^2$, $-2m^2$

**16.** $-4xy$, $-3xy$

**17.** $2.5x$, $-0.5x$

**18.** $-8.1a^2b^2$, $4.5a^2b^2$

**19.** $3x$, $-4x$, $-6x$, $5x$

**20.** $-3.2pq$, $-4.8pq$, $7.6pq$, $-0.3pq$

**21.** $\frac{3}{4}lw$, $-\frac{1}{2}lw$, $\frac{1}{5}lw$

Subtract:

**22.** $-10$ from $7$

**23.** $-4.6$ from $-5.8$

**24.** $-7x$ from $-5x$

**25.** $85$ from $-15$

**26.** $-8y^2z$ from $4y^2z$

**27.** $-3x^2$ from $2x^2$

28. $3.5a$ / $-1.5a$

29. $\frac{1}{2}x^2$ / $\frac{3}{4}x^2$

30. $-0.9ab$ / $1.2ab$

31. $3a^2-6b^2$ / $4a^2+2b^2$

32. $-7x+8y$ / $-9x-y$

33. $2x^2-7x+5$ / $x^2+3x+9$

Collect like terms in the following:

34. $2a^2-7b^2+11ab-4b^2-3ab-5a^2$.

35. $12x-17y+8x-20x+20y$.

36. $mn-5kn-4km-8mn+6kn-km$.

37. $6x^2+8x-15-5x^2-8x+12$.

38. $11y-8y+15-5y-8-9-y+2$.

39. $-r^3-8r^2+17s^3+10r^3-25s^3$.

## REVIEW SET No. 2

1. If $a = 4$ and $b = -3$, find the value of $a^2-3ab+b^2$.
2. Find the value of $m^4-3n^4+2k^4$ when $m=n=k=-1$.
3. If $x = 2$ and $y = -1$, evaluate $(x+y)/(x-y)$.
4. Evaluate $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$ when $x = \frac{1}{2}$, $y = \frac{1}{3}$, and $z = \frac{1}{4}$.
5. Evaluate $(a^2-3ab)/(4ab+b^2)$ when $a=-3$ and $b=2$.

Add:

6. $-9x^2$ / $3x^2$ / $5x^2$ / $-4x^2$

7. $1.5ab$ / $-1.5ab$ / $-4.3ab$ / $2.5ab$

8. $11z$ / $5z$ / $-9z$ / $-3z$

9. If $a = -2$, what is the value of $a^3-3a^2+2a-5$?

Subtract:

**10.** $3a^2b - 9ab^2$ / $4a^2b + 2ab^2$

**11.** $-6x^2 - 4xy + 7y^2$ / $x^2 - 5xy - 2y^2$

**12.** $2m^3 - 6m^2 + 7m - 1$ / $m^3 + 2m^2 - m + 4$

**13.** $5x^2 - 4x + 7$ / $6x^2 - 2x - 3$

**14.** Check Ex.13 when $x = 1$.

**15.** Check Ex.12 when $m = 2$.

Insert parentheses in the following:

**16.** $2x - 8y$ **17.** $ax + ay$ **18.** $3x^2 + 6x$

**19.** $ay^2 + ay$ **20.** $a^2b^2 - 3ab$ **21.** $4x^2 - 6x$

**22.** $12x^3y^3 - 4xy$ **23.** $3r^3 + 12r^2$ **24.** $6m^2n + 2mn^2 - 4m^2n^2$

Multiply:

**25.** $4a \times 3a$ **26.** $4a \times -3a$ **27.** $-4a \times 3a$

**28.** $-4a \times -3a$ **29.** $-4m^2n \times -7mn^2$ **30.** $-b^2 \times -2b^2$

**31.** $4.3x \times -2x^2$ **32.** $a^2 - 1 \times a$ **33.** $2x^2 + 4y \times -3x^2$

**34.** $6a - 2b \times 3a$ **35.** $3x + 2y \times -4x$ **36.** $-2m - 3n \times -5m$

Divide:

**37.** $8b/8b$ **38.** $12x/x$ **39.** $12x/12$

**40.** $x^2/-x$ **41.** $4m^2/2$ **42.** $-x^3/x$

**43.** $-x^3/-x^2$ **44.** $8a^2/-4a$ **45.** $-9y^3/3$

**46.** $\dfrac{5a + 5}{5}$ **47.** $\dfrac{6x^2 - 9x}{3x}$ **48.** $\dfrac{8r^2s^2 - 4rs}{-4rs}$

Simplify the following by removing parentheses and collecting like terms:

**49.** $-6(a-7)+4(5a-9)$ **50.** $4x(x+a)-a(2x-3a)$

**51.** $3x(7x-y)-5x(4x+3y)$ **52.** $(3a^2-7a+4)-(a^2+5a-5)$

## REVIEW SET No. 3

**1.** If $\frac{1}{f} = \frac{1}{p} + \frac{1}{q}$, find $f$ when $p = 3$ and $q = -4$.

**2.** Find the value of $3x^2 - 4x + 5$ when $x$ equals $0$, $-1$, $2$, $-\frac{1}{2}$, $0.3$.

Add:

**3.** $-4.3a$ / $9.2a$ **4.** $-7x^2$ / $2x^2$ **5.** $-6.1xy$ / $-2.4xy$

**6.** $11m^2n^3$ / $-5m^2n^3$ **7.** $-10y^2$ / $-4y^2$ **8.** $7(x+y)$ / $-2(x+y)$

**9.** $-3(a-b)$ / $-4(a-b)$ **10.** $0.3a^2$ / $-2.7a^2$ **11.** $4(m+n)$ / $-5(m+n)$

Subtract:

**12.** $9ab$ / $-2ab$ **13.** $7.5x$ / $4.5x$ **14.** $-10x^2y$ / $12x^2y$

**15.** $-7x$ / $4x$ **16.** $18xyz$ / $-19xyz$ **17.** $-2(x+y)$ / $-5(x+y)$

**18.** $-2.4t$ / $3.8t$ **19.** $-7a^3b^3$ / $9a^3b^3$ **20.** $-11m^2n$ / $-4m^2n$

Multiply:

**21.** $3x$ / $-4x$ **22.** $8xy$ / $2y$ **23.** $-5x^2$ / $-6x$

**24.** $\frac{1}{2}ab$
$-\frac{3}{4}a^2b^2$

**25.** $-2.4a^2b$
$-0.5ab^2$

**26.** $3x^2$
$-8xy^3$

**27.** $2r-3s$
$-4r$

**28.** $a+2b$
$2a-b$

**29.** $3x-4y$
$x+2y$

Divide:

**30.** $-5x^2$ by $5x$.

**31.** $2.5xyz$ by $-0.5yz$.

**32.** $-18ab$ by $-6a$.

**33.** $-45k^3$ by $15k^2$.

**34.** $-a^3b^3$ by $2ab$.

**35.** $44pq$ by $-11p$.

**36.** $-7x^2$ by $-x^2$.

**37.** $2x^2-4xy$ by $2x$.

**38.** $-6a^2b+15ab^2$ by $-3ab$.

**39.** $20x^3+40x^2$ by $-20x^2$.

**40.** $m^2-m$ by $m$.

**41.** $-18r^3t^2-45r^2t^3$ by $-9r^2t^2$

Combine into a single fraction:

**42.** $\frac{3}{7}+\frac{2}{7}$

**43.** $\frac{3}{x}+\frac{2}{x}$

**44.** $\frac{9}{20}-\frac{3}{10}$

**45.** $\frac{9}{2a}-\frac{3}{a}$

**46.** $\frac{1}{6}+\frac{1}{9}$

**47.** $\frac{1}{ab}+\frac{1}{bc}$

**48.** $\frac{4}{3a}-\frac{3}{4a}$

**49.** $\frac{5}{x^2}+\frac{1}{x}$

**50.** $\frac{11}{24}-\frac{7}{16}$

Reduce to lowest terms:

**51.** $\frac{(3x)\ (-2a)\ (-5b)}{-10ax}$

**52.** $\frac{(-x^2)\ (7xy)\ (3y^2)}{-21x^3y^3}$

**53.** $\frac{(4m^3)\ (-5km)\ (-3kn)}{12km^3}$

**54.** $\frac{-125x^3y^4z^5}{50x^3y^3w}$

Simplify by removing parentheses and collecting like terms:

55. $6(2x - 7) - 5(x - 8) + 2(x + 1)$

56. $(2k + m - n) - (3k + 2m - 5n) + 4(k - m - n)$

## REVIEW SET No. 4: FORMULAS

For each of the following formulas, find the value of the unknown number. Take $\pi = 22/7$.

1. $V = \pi r^2 h$; $r = 7$, $h = 10$.

2. $A = \frac{1}{2}h\,(a + b)$; $a = 12$, $b = 8$, $h = 2.4$.

3. $s = 50t - 16t^2$; $t = \frac{3}{4}$.

4. $C = \frac{5}{9}\,(F - 32)$; $F = 212$.

5. $S = n\,(a + l)/2$; $n = 100$, $a = 1$, $l = 100$.

6. $D = b^2 - 4ac$; $b = 1.5$, $a = 2$, $c = -1$.

Make a formula for each of the following rules, giving the units associated with your formula.

7. The volume of a ball is $4\pi/3$ times the cube of its radius.

8. The distance traversed equals the average rate of speed times the time taken. (More exactly, the rule should be stated thus: the number of units in the distance traversed equals the number of units in the average rate of speed times the number of units of time taken.)

9. The simple interest equals the principal times the rate times the time.

10. The distance around the rim of a wheel equals $2\pi$ times its radius.

11. The area of the tin in a length of stove-pipe equals its circumference times its length.

The 3 tables which follow show relations between pairs o numbers. For each table, write a formula for the relationship

**12.**

| $x$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $y$ | 3 | 6 | 9 | 12 | 15 | 18 | 21 | 24 |

**13.**

| *Number of inches* ($i$) | 2 | 3 | 5 | 10 | 50 | 100 |
|---|---|---|---|---|---|---|
| *Number of centimeters* ($c$) | 5 | 7.5 | 12.5 | 25 | 125 | 250 |

**14.**

| *Father's age in years* ($f$) | 32 | 37 | 42 | 47 | 52 | 57 |
|---|---|---|---|---|---|---|
| *Son's age in years* ($s$) | 8 | 13 | 18 | 23 | 28 | 33 |

**15.** Solve the formula of Ex. 1 for $h$ and for $r$.

**16.** Solve the formula for Ex. 2 for $h$.

**17.** Solve the formula of Ex. 4 for $F$.

**18.** Solve the formula of Ex. 5 for $n$.

**19.** Re-arrange the formula $P = k/V$ so that $V$ is th subject.

**20.** Solve the formula $R = aS + b$ for $S$.

**21.** The volume of a tin can is given by $V = \pi r^2 h$. Wha is the effect on the volume ($a$) when $r$ is doubled? ($b$) when is halved and $h$ is doubled? ($c$) when both $r$ and $h$ are doubled

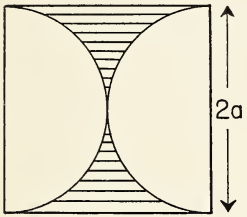
**22.** The average rate of speed is given by the formula $r = d/t$. What is the effect on $r$ when ($a$) $d$ is doubled? ($b$) $t$ is doubled? ($c$) $d$ is doubled and $t$ is halved?

**23.** The volume of a spherical tank is given approximately by the formula $V = 4.19r^3$. What change is produced in $V$ by ($a$) doubling $r$? ($b$) halving $r$? ($c$) tripling $r$?

**24.** In the formula $V = lwh$, what is the effect on $V$ when *a*) $l$ and $w$ are doubled? (*b*) $w$ is doubled and $h$ is tripled? *c*) $l$ is halved, $w$ is halved, and $h$ is increased 4-fold?

Make formulas for the area and the boundary of each of the haded regions in the figures following:

25.

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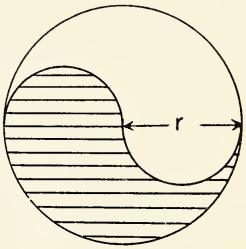


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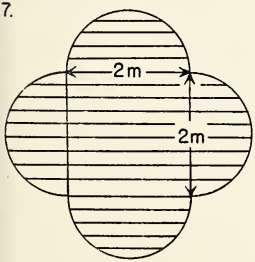


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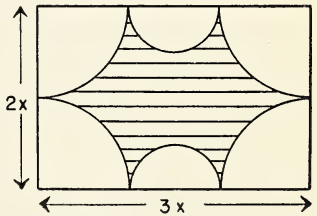



## REVIEW SET No. 5: EQUATIONS

Solve for $x$ and verify:

1. $x+5=7$
2. $x-5=7$
3. $5x=7$
4. $\frac{1}{5}x=7$
5. $x+15=11$
6. $x-4=-6$

7. $8x = -3$
8. $x/3 = -2$
9. $2x+3=7$
10. $3x-2=7$
11. $\frac{1}{2}x+5=4$
12. $\frac{1}{3}x-3=2$
13. $11=5x-4$
14. $4x+8=-12$
15. $\frac{3}{4}x=-6$
16. $1.5x=0.45$
17. $ax=b$
18. $x+a=b$
19. $0.2x+1=2$
20. $\frac{2}{3}x+3=3\frac{2}{3}$
21. $x+\frac{3}{4}=\frac{5}{4}$
22. $\frac{3}{5}x+0.2=1$
23. $6x-6.2=12.4$
24. $4.3x+2.4=-6.2$
25. $\frac{2x}{3}+8=10$
26. $\frac{3}{4}x-0.5=2.5$
27. $2x=x+8$
28. $0.5x+4=x$
29. $4x+5=6x+3$
30. $17x+18=2$[illegible]$x$
31. $7x-2=5x+1$
32. $10x=6x-8$
33. $5x-3=17+3x$
34. $\frac{3}{2}x+9=3x-6$
35. $\frac{x}{10}+23=3$
36. $\frac{x-2}{3}=4$
37. $4.2x+1.6=5.8$
38. $x/3+20=x+4$
39. $-3x-7=8+2x$
40. $2.5x-(x-3)=\frac{3}{2}(4-x)-4$
41. $5x-2(x-3)=3(4-x)-9$
42. $\frac{2x+1}{3}=\frac{2x+3}{4}$
43. $\frac{1}{2}(7x-5)=\frac{2}{3}(8x-23)$
44. $x/2-3=x/3+1$
45. $8-3(x-1)=14$
46. $3x-10=\frac{1}{2}x-15$

## REVIEW SET No. 6: PROBLEMS

If $x$ represents an unknown number, translate the followin phrases and sentences into algebra.

1. An unknown number increased by 6.

2. 11 is added to an unknown number.

3. An unknown number decreased by 4.

4. An unknown number doubled.

5. Half of an unknown number.

6. An unknown number is tripled and the result increased by 15.

7. An unknown number is increased by 15 and the result tripled.

8. An unknown number is cubed.

9. 7 times the square of an unknown number.

10. One quarter of an unknown number is increased by 9.

11. Five times an unknown number equals the number increased by 8.

12. If 5 is added to an unknown number, the result is 12.

13. An unknown number is doubled when 6 is added.

14. Three times the square of an unknown number equals 75.

15. 5 less than twice an unknown number equals 7 more than 3 times the number.

* * *

16. If a certain number is doubled and then 3 is added to the result, the sum is 33. Find the number.

17. If a number is tripled and the result decreased by 15, the remainder is 60. Find the number.

18. One of two numbers is five times the other and their sum is 72. Find the numbers.

19. A pole 32 feet long is cut into two parts so that the larger part is three times as long as the smaller. How long is the smaller part?

**20.** After the number of students in a class increased by a quarter, it was found to have 35 students. What was the number in the class before the increase?

**21.** The length of a rectangular field is twice the width. If the perimeter is 144 rods, find the dimensions.

**22.** Divide \$360 into three parts so that the second part is twice the first, and the third part three times the first.

**23.** Mr. Jones is 24 years older than his son, John. If the sum of their ages five years from now will be 52 years, find John's present age.

**24.** Mary saved 4 times as much money as Louise and together they saved \$6.20. How much did Mary save?

**25.** A suit of clothes is sold at \$49, which is 40% above cost. Find the cost price of the suit.

**26.** What sum of money invested at 3% per annum will yield interest amounting to \$60.90 in one year?

**27.** A rope 250 feet long is cut into two pieces so that the smaller piece is 3/7 of the larger. How long is the larger piece?

**28.** One angle of a triangle is 20° more than the smallest angle and the third angle is 34° more than the smallest angle. Find the angles.

**29.** One angle of a triangle is three times the smallest angle and the third angle is 12° more than double the smallest angle. Find the angles.

**30.** Find two consecutive numbers whose sum is 59.

**31.** 500 tickets were sold for the football game, giving total gate receipts of \$105. If some of the tickets were sold at 15 cents and the rest at 25 cents, find the number of each kind sold.

**32.** Find three consecutive numbers whose sum is 129.

**33.** A taxi-cab charges 20 cents for the first quarter-mile and 10 cents for each quarter-mile thereafter. If Mr. Simpkins owed the driver $1.90, how far did he ride?

**34.** The perimeter of an isosceles triangle is 44 feet, and its base is 10 feet less than the length of one of the two equal sides. Find the lengths of the sides.

CHAPTER II

# EQUATIONS WITH TWO UNKNOWNS

*"The animals came in two by two:*
*These bones shall rise again."*

Author Unknown

## *Something More About Equations*

We have made use of the fact that a few basic principl readily enable us to find an unknown number in some equation For example, we can

***(i) add a given number to each side of an equation,***

***(ii) subtract a given number from each side of an equatio***

***(iii) multiply each side of an equation by a given numbe***

***(iv) divide each side of an equation by a given number, n zero.***

*These operations preserve the equality and at the same time allo us to get the unknown number alone on one side.*

You know by now, of course, that many problems requir the finding of an unknown number, since numbers are use whenever we count or measure. Our general plan for findin an unknown number is to let it be $x$, then translate the facts the problem into an algebraic equation, and, finally, solve th equation for $x$.

What if there are ***two*** unknown numbers in the problem Will the foregoing method work, or must we devise a new one An example will throw some light on these questions.

*Example.* Farmer Fields feeds his horses on a mixture shelled corn and oats. He finds that 10 bushels of the mixtur

eigh 494 lb. and wonders how many bushels of each cereal
ıe mixture contains. His son, Fred, knows that a bushel of
ats weighs 34 lb. and one of shelled corn weighs 56 lb. Fred
thus able to answer his father's question.

Here is how Fred did it. He let the number of bushels of oats be $x$.

Then the number of bushels of corn was $10 - x$. Therefore,

the weight of oats was $34x$ lb., and

the weight of corn was $56\ (10 - x)$ lb.

This gave Fred the equation

$$34x + 56\ (10 - x) = 494.$$

It follows that

$$34x + 560 - 56x = 494,$$

and $$-\ 22x = -\ 66,$$

and $$x = 3.$$

The mixture contained 3 bushels of oats and 7 bushels of corn.

Notice that although this problem involves ***two*** unknown umbers, Fred was able to solve it by using only one unknown umber $x$. What would be the result if we used two unknown umbers? Let us see.

Suppose the mixture contains $x$ bushels of oats and $y$ ushels of shelled corn. Then $x + y$ must equal 10. Moreover, since the oats weigh $34x$ lb. and the corn weighs $56y$ ., we have

$$34x + 56y = 494,$$

s well as $$x + y = 10.$$

***Instead of getting one equation with one unknown umber, we have two equations involving two unknown umbers.*** The question at once arises as to whether we can olve such equations. In other words, is it possible to find alues for $x$ and $y$ that will satisfy ***both*** equations? Of course, he answer is *yes* by virtue of the first solution to the problem. Try to find a method of solving the two equations.

* * *

Such equations as these are said to be ***equations of t first degree in x and y.*** You see that the equations invol only the first power of $x$ and of $y$. The two equatio make up a ***system*** in $x$ and $y$.

## *How to Solve Two Equations of the First Degree in x and y*

In order to arrive at some method of solving two equatio of the first degree in $x$ and $y$, let us examine some simp examples. Consider the system:

$$x + y = 4$$
$$x - y = 2.$$

If $x$ and $y$ stand for the same numbers in both equations, c you suggest a method of getting rid of one of the unknow numbers?

* * *

Yes, if we add the two equations, we have

$$2x = 6, \text{ whence } x = 3.$$

Substitution shows that $y = 1$ and that the values $x =$ $y = 1$, satisfy both equations of the system. We call the pa of values, $x = 3$ and $y = 1$, a ***solution*** of the system.

Observe that our success was due to the fact that w ***eliminated one of the unknowns.*** This left us with o equation containing the single unknown $x$ — an old friend.

An important question comes up here: What right hav we to add the two equations?

* * *

The answer involves one of the basic assumptions mathematics:

***When equal numbers are added to equal numbers, the sums are equal.***

hat other assumption would you make as a partner to the e above?

Now try to eliminate one of the unknowns in each of the lowing systems:

) . . . $x+2y=5$ $x+y=3$. (*b*) . . . $2x+3y=7$ $x-3y=-1$. (*c*) . . . $2x+3y=7$ $x-y=1$.

* * *

In system (*a*), subtract the equations, getting $y = 2$. bstitution then gives $x = 1$. What basic assumption stifies your subtracting these equations? By adding the uations of system (*b*), you eliminate $y$ and get $3x = 6$, nence $x = 2$ and $y = 1$.

System (*c*) requires something extra since neither addition r subtraction will eliminate an unknown from the equations they stand. Do you see a way out of the difficulty?

* * *

The second equation is equivalent to

$$3x - 3y = 3. \quad \text{(Why?)}$$

ld this equation to the first one and get $5x = 10$, whence $= 2$ and $y = 1$. How might $x$ have been eliminated instead $y$?

The following examples further illustrate the ***method of imination.*** Examples like 2 and 3 afford you some good actice in the skills of arithmetic.

*xample* 1. Solve the system: $3x - 4y = 8$ . . . (1)
$2x + 3y = -6$ . . . (2).

NOTE: A common way of referring to equations is by numb which are placed in parentheses beside the equatio

In order to eliminate $x$, we multiply (1) by 2 and (2) by

This gives

$$6x - 8y = 16 \quad \ldots \quad (3)$$
$$6x + 9y = -18 \quad \ldots \quad (4).$$

Subtracting, we have

$$-17y = 34,$$

and

$$y = -2.$$

Substituting for $y$ in (1), we obtain

$$3x + 8 = 8,$$

and

$$x = 0.$$

*Verification*: In order to verify, we must show that the values satisfy the ***given*** equations. By substitution, we get

Left side of (1) $= 0 - 4(-2) = 8 =$ right side of (
Left side of (2) $= 0 + 3(-2) = -6 =$ right side of (

*Question*: Why should the values for $x$ and $y$ be substitut in the given equations rather than in (3) and (4)?

### EXERCISES

Solve the following systems:

1. $x + y = 11$
   $x - y = 3.$

2. $x - 3y = 1$
   $2x + 3y = 29.$

3. $5x + 3y = 21$
   $5x - 2y = 11.$

4. $x - 4y = -9$
   $2x - 4y = -14.$

5. $6x - 5y = 17$
   $x + 5y = -3.$

6. $x + 5y = 1$
   $x - 2y = 8.$

7. $2x + 3y = 14$
   $x + y = 5.$

8. $3x - 4y = 1$
   $5x + 2y = 6.$

9. $4x + 5y = 2$
   $5x + 4y = 7.$

10. $7x - 3y = 5$
    $5x - 2y = 4.$

*Example* 2. Solve the system: $\frac{1}{2}x - \frac{1}{4}y = \frac{1}{6}$ . . . (1)
$x + \frac{1}{2}y = \frac{2}{3}$ . . . (2).

*Method* 1. Let us first solve the system with the fractions left in. We can eliminate either $x$ or $y$ by multiplying (2) by $\frac{1}{2}$. This gives

$$\frac{1}{2}x - \frac{1}{4}y = \frac{1}{6} \quad . \ . \ . \quad (3)$$
$$\frac{1}{2}x + \frac{1}{4}y = \frac{1}{3} \quad . \ . \ . \quad (4).$$

Now add (3) and (4). We have

$$x = \frac{1}{6} + \frac{1}{3} = \frac{1}{2}.$$

Substituting in (2), we get

$$\frac{1}{2} + \frac{1}{2}y = \frac{2}{3},$$

whence $\frac{1}{2}y = \frac{1}{6}$, and $y = \frac{1}{3}$.

*Verification*:

Left side of (1) $= \frac{1}{4} - \frac{1}{12} = \frac{1}{6}$ = right side of (1).
Left side of (2) $= \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$ = right side of (2).

*Question*: How would you eliminate $x$ from (3) and (4)?

*Method* 2. As our first step, we might clear the fractions from (1) and (2). Multiplying both sides of (1) by 12, and both sides of (2) by 6, we have

$$6x - 3y = 2 \quad . \ . \ . \quad (5)$$
$$6x + 3y = 4 \quad . \ . \ . \quad (6).$$

The completion is left as an exercise.

*Example* 3. Solve the system:

$$0.4x - 0.5y = -0.17 \quad . \ . \ . \quad (1)$$
$$0.5x + 2y = 1.1 \quad . \ . \ . \quad (2).$$

*Method* 1. Let us leave the decimal fractions in the equations. We can eliminate $y$ by multiplying (1) by 4 and then adding. This gives

$$1.6x - 2y = -0.68$$
$$0.5x + 2y = 1.1$$
$$2.1x = 0.42$$
$$x = 0.42/2.1 = 0.2.$$

Substitution in (2) gives

$$0.1 + 2y = 1.1$$

whence $2y = 1$ and $y = 0.5$.

The verification is left as an exercise.

*Method* 2. The decimal fractions can be cleared away multiplying (1) by 100 and (2) by 10. The details are l as an exercise.

### EXERCISES

1. Complete the solution of Example 2 using Method 2.
2. Verify the answers obtained in Example 3.
3. Complete the solution of Example 3 using Method 2.

Solve the following systems; verify your solutions.

4. $3x - 5y = 0.5$
   $2x - y = 1.5$.

5. $x + \frac{1}{2}y = 4$
   $2x - y = 4$.

6. $0.3x + 0.2y = 0.1$
   $0.2x - 0.3y = 0.5$.

7. $\frac{3}{4}x - 2y = 2$
   $x + 8y = 24$.

8. $x/3 + y/5 = 3$
   $x/2 + y = 8$.

9. $2.4x - 1.6y = 1.04$
   $0.8x - 2.5y = 0.74$.

10. $\frac{3}{2}x + \frac{5}{6}y = \frac{1}{3}$
    $x - \frac{2}{3}y = \frac{5}{6}$.

11. $5x + 1.3y = 8.7$
    $3x - 2.5y = 8.5$.

12. $3x/2 + 7y = -1$
    $x + 2y = 2$.

13. $1.8x - 4.3y = 1.58$
    $0.9x + 0.8y = 0.2$.

14. $12x - 3.8y = 4.4$
    $3x + 2y = 7$.

15. $4x - 3y = 0$
    $\frac{1}{2}x + \frac{1}{4}y = \frac{2}{2}\frac{5}{4}$.

16. $2x/7 + 5y/6 = 13/42$
    $x/7 - 10y = -27/14$.

17. $0.5x - 3.1y = -10.3$
    $2.5x + 4.5y = 8.5$.

## *An Alternative Method of Solution*

The linear systems in the last three examples were solve

y the method of elimination. The following example illustates an alternative method.

*xample* 4. Solve the system: $x - 2y = 5$ . . . (1)
$5x + 3y = -1$ . . . (2).

From (1), we get $x = 2y + 5$. If $x$ and $y$ stand for the same numbers in both equations, we can substitute $2y + 5$ for $x$ in (2). This gives

$$5(2y + 5) + 3y = -1,$$

whence $13y = -26$, and $y = -2$.
Also $x = 2y + 5 = -4 + 5 = 1$.
Verify the solution $x = 1$, $y = -2$.

The foregoing example illustrates the method of ***substitution.*** The method consists of solving one equation for $x$ $y$, and then substituting the value of $x$ or $y$ thus obtained in e second equation. In more advanced work in algebra, you ll find that more complicated systems can be dealt with by e same device.

### EXERCISES

Solve the following systems by substitution and verify your swers.

1. $x + y = 11$
   $x - y = 3.$

2. $x - 3y = 1$
   $2x + 3y = 29.$

3. $x - 4y = -9$
   $2x - 4y = -14.$

4. $6x - 5y = 17$
   $x + 5y = -3.$

5. $2x + 3y = 14$
   $x + y = 5.$

6. $3x - 4y = 1$
   $5x + 2y = 6.$

7. $4x + 5y = 2$
   $5x + 4y = 7.$

8. $7x - 3y = 5$
   $5x - 2y = 4.$

## *Problems Involving Two Unknowns*

Let us now go back to Farmer Fields' problem on page 21. appears that the problem can be solved by introducing either

one unknown or two. In the first case, we must construct
equation; in the second case, the solution leads to a system
two equations of the first degree in $x$ and $y$. Which is
better method? The answer depends on the nature of
problem; some problems lend themselves readily to the use
one unknown, others are more conveniently handled with t

Since you should be familiar with both methods, it will
wise to get some practice with the new one. The gen
method of attack is the same with either one or two unknow
For convenience, we recall the seven suggested steps
attacking a problem as worked out in Book I.

(1) *Read the problem carefully to find out what is needed.* (Sometimes a diagram is helpful in picturing the fac

(2) *Let some letter, say x, stand for an unknown number. If there are two unknown numbers, introduce, if desi a second letter for the other unknown number.*

(3) *Using the letter-number(s), translate the English st ments into algebraic expressions.*

(4) *Re-read the problem and find what expressions are eq*

(5) *Form an equation from the equal expressions; if unknowns are used, form two equations.*

(6) *Solve the equation or the system, as the case may be.*

(7) *Test whether the answer satisfies the conditions of problem.*

*Example.* Tom Tourist drove his car a distance of 240 m in 7 hours. Part of the trip was over paved road on which averaged 40 m.p.h., and part was on gravel on which averaged 30 m.p.h. How many hours did he drive on pa road and how many on gravel?

The problem can be solved using two unknowns as follo

Let Tom travel $x$ hours on paved road and $y$ hours on gravel. Since the total time of travel is 7 hours, we have

$$x + y = 7 \quad . \quad . \quad . \quad (1).$$

Now the number of miles travelled equals the number of m.p.h. in the average speed times the number of hours taken. (What formula gives us this information?) Therefore, the distance travelled on pavement is $40x$ miles, and the distance on gravel, $30y$ miles. Since the total distance travelled is 240 miles, it follows that

$$40x + 30y = 240 \quad . \quad . \quad . \quad (2).$$

(1) gives $30x + 30y = 210$

Therefore $10x = 30$, and $x = 3$.

Hence, Tom travelled 3 hours on pavement and 4 hours on gravel road.

*Verification*: The time taken is 7 hours which checks with the problem. The distance covered is 120 miles on pavement and 120 miles on gravel, which gives the correct total of 240 miles for the trip.

## EXERCISES

Solve each of the following problems by introducing two nknowns:

**1.** The sum of two numbers is 37 and their difference is 3. ind the numbers.

**2.** The sum of two numbers is 9. If the first number is oubled and the result increased by 3 times the second number, ıe sum is 22. Find the numbers.

**3.** The difference of two numbers is 6. The sum of twice ıe first number and 5 times the second is 15. Find the umbers.

**4.** Find two numbers such that three times the first in creased by four times the second is 90, and the first diminishe by twice the second is $-20$.

**5.** The perimeter of a rectangular field is 120 rods. Th sum of the length and twice the width is 85 rods. Find th length and width.

**6.** Mr. Filbert wishes to mix some nuts worth 40 cents pound with others worth 80 cents a pound so as to get 10 pounds of mixture worth 48 cents a pound. How man pounds of each kind of nuts should be mixed?

**7.** A pole 32 feet long is cut into two parts so that th larger part is three times as long as the smaller. How lon are the parts?

**8.** Mr. Affluent divides \$250 between his two childre John and Mary. If John's share increased by twice Mary share is equal to \$365, find the amount of money given to eac child.

**9.** The sum of two numbers is 175. If four times th smaller equals three times the larger, find the numbers.

**10.** Stanley went to a post-office to buy five dollars' wort of 3-cent and 4-cent stamps. If he brought home 150 stamp how many of each did he buy?

**11.** Mr. Simpkins travelled from Stoneville to Rockburg, distance of 205 miles, in 6.5 hours. Part of the trip was mad on good road over which he averaged 35 m.p.h., and the re of the trip was made at an average of 20 m.p.h., because th road was under construction. How many miles of the roa were under construction?

**12.** The Tigers Club of Arcadia purchased 1000 pounds candy for distribution to children at Christmas. Some the candy cost 30 cents a pound and the rest cost 40 cents pound. If the total cost was \$335, how many pounds of eac kind were bought?

**13.** A box contains 140 coins having a total value of $8.75. f the coins are all either nickels or dimes, how many of each ind are there?

**14.** The sum of the terms of a fraction is 12. If five times he numerator is decreased by three times the denominator, he difference is 4. What is the numerator of the fraction?

**15.** Work Exercise 14, using one unknown.

## *It All Depends*

Let us consider the two equations

$$x + 3 = 4 \quad . \quad . \quad . \quad (a)$$

nd

$$x + y = 4 \quad . \quad . \quad . \quad (b).$$

'he role played by $x$ in equation $(a)$ is quite different from that layed in $(b)$. In $(a)$, $x$ stands for the number 1 and no other. 'he equation $(b)$, on the other hand, says that the sum of the air of numbers $x$ and $y$ is 4. What sort of condition does this lace on $x$? Can $x$ have more than one value?

* * *

Now try to guess a pair of numbers that satisfy the con- ition $(b)$. Many such pairs will occur to you. For example, ou might have $x = 2$, $y = 2$; $x = 5$, $y = -1$; $x = 0$, $y = 4$; $= 1.5$, $y = 2.5$; and so on. How many such pairs can be btained? * * * Of course, the number of such airs is unlimited; you could spend the rest of your life writing iem and still not begin to exhaust the supply. Let us arrange a table some pairs of numbers whose sum is 4.

$$x + y = 4$$

| $x$ | $-871$ | $-50$ | $-4$ | $-3$ | $-2\frac{3}{4}$ | $-2$ | $-1$ | 0 | 1 | 2 | 3 .5 | 4 | 5 | 6 | 100 | 1004.5 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $y$ | 875 | 54 | 8 | 7 | $6\frac{3}{4}$ | 6 | 5 | 4 | 3 | 2 | 0 .5 | 0 | $-1$ | $-2$ | $-96$ | $-1000.5$ |

You see that we can give to $x$ any value we please and always nd a value of $y$ to match it; that is, a $y$ which combines with $x$

to give the sum 4. In the same way, we can give $y$ any valu and then find an $x$ to pair with it.

> *Letter-numbers like $x$ and $y$ which stand for any one of a set of numbers are called* ***variables.***

You should note carefully that *while we can give $x$ any valu or $y$ any value, we cannot give* ***both*** *$x$ and $y$ any values.* Th pair of values must be chosen so as to satisfy the condition i equation $(b)$. Thus $x = 4$ and $y = -17$ would not b accepted; this pair does not satisfy $(b)$. In other words, w can assign a value to *one* variable only; then $(b)$ fixes the valu of the other variable which matches. This means that *on variable is* ***related to*** *or* ***depends on*** *the other.* The exa nature of the relation or dependence is determined by $(b)$.

> *In the equation $x + y = 4$,*
>
> (1) *$x$ and $y$ are* ***variables*** *since each may represent any one of an unlimited set of numbers;*
>
> (2) *$x$ and $y$ are* ***related*** *since the value of each depends on that of the other.*

*Example.* A study of the table on page 31 discloses son interesting facts about the relation between $x$ and $y$.

$(a)$ By scanning the table from left to right, we see th as $x$ increases, $y$ decreases.

$(b)$ A change of 1 unit in $x$ always produces a change one unit in $y$. Similarly, a change of $n$ units in produces a change of $n$ units in $y$.

### EXERCISES

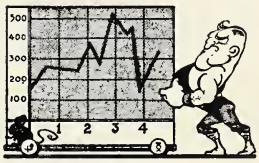
1. Make a table by finding about 15 pairs of values whi

atisfy the relation $x - y = 4$. Use the table to answer the llowing: How does $y$ change as $x$ increases? How does $x$ nange as $y$ increases? Does a change of 1 unit in $x$ always roduce a change of 1 unit in $y$?

**2.** Make a table like that on page 31 for the equation $x = y + 6$. Use the table to answer the following: How oes $y$ change as $x$ increases? As $x$ decreases? Does a unit nange in $x$ produce a unit change in $y$? Does a unit change in always produce the same change in $y$?

**3.** Make a table for the equation $3x + y = 4$. Use the ble to answer the following: How does $y$ change as $x$ increases? ow does $x$ change as $y$ increases? What change is produced $y$ by an increase of 1 unit in $x$?; by an increase of 2 units in $x$?

## *Re-enter the Graph*




In Book I, we learned that a graph is a vivid way in which picture a relation between numbers. Since we shall make se of graphs on a number of occasions in this book, we had tter recall some of the facts about them.

In order to make a graph showing the dependence of two riables such as $x$ and $y$, the following are necessary:

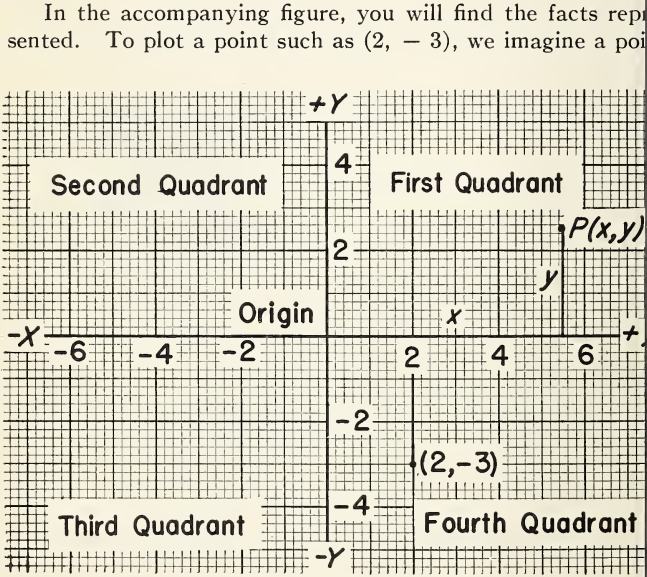
(1) a pair of perpendicular lines, one for the axis of $x$ and one for the axis of $y$;

(2) a unit of length with which to represent numbers on the axes.

he axes and units may be conveniently placed on graph per which is ruled into squares of various sizes, usually with les of 1 inch, $\frac{1}{2}$ inch, and 1/10 inch.

It is customary to call the horizontal axis the ***axis of x*** a the vertical axis, the ***axis of y.*** The point of intersection the two axes is the ***origin.*** Remember that ***positive*** numbe are laid off from the origin to the ***right*** on the $x$-axis, ***negati*** numbers to the ***left;*** on the $y$-axis, ***positive*** numbers are l off ***above*** the $x$-axis starting from the origin, ***negati*** numbers are laid off ***below*** the $x$-axis. The two numbers th locate a point on the graph are called the ***coordinates*** of t point. We write them in parentheses thus: (2, − 3), whe the first number is always understood to be the $x$-coordina and the second number, the $y$-coordinate of the point.

In the accompanying figure, you will find the facts rep sented. To plot a point such as (2, − 3), we imagine a poi




to start at the origin and move 2 units to the right along t $x$-axis, then 3 units downwards parallel to the $y$-axis as show

### EXERCISES

The following exercises should be done on graph paper. se one-half inch as a unit; label the axes properly.

1. Plot the following points: (5, 4), (− 3, 2), (4, 0), − 4, − 6), (0, 5), (3, − 4), (− 6, 6), (− 2, 0), (3.4, − 5), , − 1), (− 4.6, − 2.8).

2. Name the quadrant in which each point of Ex. 1 lies.

3. Under what condition or conditions will the point $(x, y)$ : (*a*) in the first quadrant? (*b*) in the second quadrant? ) in the third? (*d*) in the fourth? (*e*) on the $x$-axis? (*f*) the $y$-axis?

4. One vertex of a square with sides 5 units long is at the igin and two of its sides lie along the positive axes. Find e coordinates of the four vertices of the square.

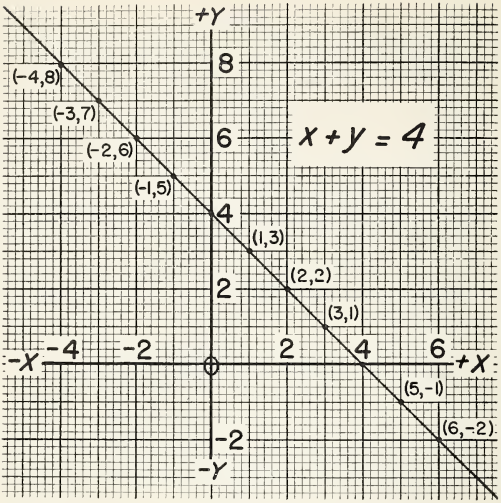
5. Plot the points (− 1, 1), (− 5, 1), and (− 1, 4). What e the coordinates of the point which, with these three points, akes up the vertices of a rectangle?

6. Find the coordinates of the point (*a*) 5 units above (4, 2), ) 3 units to the right of (− 2, 3), (*c*) 7 units below (− 3, −1), ) 1 unit to the left of (1, 0).

7. A rectangle with sides 6 units and 4 units respectively placed with one vertex at the origin and two adjacent sides ong the axes. If the rectangle lies in the third quadrant with e longer side along the $x$-axis, find the coordinates of its rtices and the midpoint of its diagonals.

## *The Graph of a First Degree Equation*

The table of values on page 31 gives us some number airs that satisfy the equation $x + y = 4$. Each of these pairs ll give us a point on a graph if we use the $x$ as an $x$-coordinate id the $y$ as a $y$-coordinate. Let us make a graph and plot many pairs as possible from the table. The result is shown the accompanying figure.




You note at once that all the points *appear* to be on straight line. Moreover, if we pick any point on this line a get its coordinates, we find their sum is 4. That is, the t coordinates of a point on the line satisfy the equation. N select a point off the line and see whether its coordina satisfy the condition. Repeat the investigation for a numb of points off the line. Do you think that the coordinates points off the line satisfy the equation? * * *

The question now arises: Do other first degree equatio in $x$ and $y$ give similar results when plotted? * * Some more examples may help you to find an answer.

### EXERCISES

For each of the following equations construct a table about a dozen number pairs which satisfy it. Then obta the graph of each equation by plotting as in the foregoi example.

1. $x + y = 6$.
2. $x - y = 3$.
3. $x = y$.
4. $2x + y = 4$.
5. $x - 2y = 2$.
6. $2x = y + 6$.
7. $3x + y = 4$.
8. $x + 2y = 2$.
9. $x - 3y = 0$.

## *Some Facts*

Of course, we cannot expect to prove general theorems from he results of a few special cases. Thus, if you observe that uring a certain time all the cars on highway No. 99 are going n the same direction, you cannot conclude that it is a one-way oad. One must be careful about making rash generalizations. 'ou are therefore asked to accept the following facts which an be proved by more advanced methods. Your experience 'ill doubtless convince you of their truth.

*THEOREM.* ***The graph of an equation of the first degree in x and y is always a straight line.***

This gives us two important ideas:

(1) *Every pair of values of x and y that satisfy the equation, gives us a point (x, y) on a straight line.*

(2) *Every point on this straight line has coordinates that satisfy the equation.*

Now you see why a first degree equation in $x$ and $y$ is called a ***linear equation*** and two such equations make up a ***linear system.***

Besides regarding the graph as being made up of plotted points, we may look at it from another point of view. If we think of $x$ and $y$ as variables, then ***the point (x, y) is a variable or moving point which changes its position as x and y change.*** From this point of view, ***the graph is the path traced by the moving point (x, y) as it moves in obedience to some condition.***

*The path traced by a moving point as it moves in accordance with some law or condition is called the* **locus** *of the moving point.* (Latin: *locus*, place)

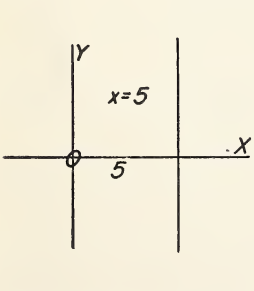
For example, the graph of $x + y = 4$ may be regarded a the *locus* (plural: *loci*) of the point $(x, y)$ as it moves so a always to keep the sum of its coordinates equal to 4.

Now, since you know that the graph of every linear equati in $x$ and $y$ is a straight line, you do not need to construct s large a table of values as that on page 31. How man pairs of values are necessary? How many pairs would it l wise to get?

* * *

It is a good plan to plot three pairs of values that satisf the equation. One pair then serves as a check. Why is preferable to plot points that are far apart on the graph Why are whole numbers preferred as coordinates?

* * *




*Example* 1. What is the graph o $x = 5$?

The equation $x = 5$ is satis fied by $x = 5$, $y = 1$; $x =$ $y = -57$; $x = 5, y = 2$; and so on Since $y$ does not occur in th equation, its values are not re stricted in any way. You migh think of the equation as

$$x + 0y = 5.$$

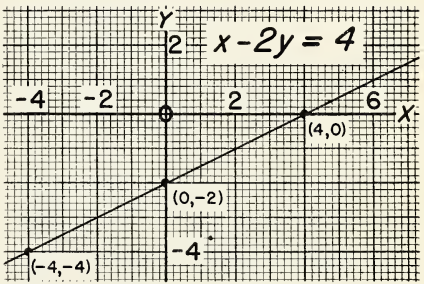
Plotting shows that all points which satisfy the equation lie o a line parallel to the $y$-axis and 5 units to the right of it.

If you regard the graph as a locus, that is, the path traced y the point $(x, y)$ as it moves so as to keep its $x$-coordinate qual to 5, then you quickly arrive at the same result.

*Example* 2. Plot the graph of $x - 2y = 4$. Use the graph o answer the following questions: (1) What happens to $y$ as $x$ ıcreases? (2) What change is produced in $y$ by an increase f 1 in $x$? (3) What change is produced in $x$ by a decrease of in $y$? (4) What is the value of $y$ when $x = 6$? (5) What is he statement in words of the condition satisfied by the moving oint $(x, y)$ as it traces this graph?

$x - 2y = 4$

| $x$ | $-4$ | $0$ | $4$ |
|---|---|---|---|
| $y$ | $-4$ | $-2$ | $0$ |




Ve make a table with three pairs of values which satisfy the quation. One of the pairs enables us to check the straight ne plotted on the graph paper.

(1) Imagine a point moving along the plotted line from ɛft to right. Since the point is moving to the right its $x$-coordinate is increasing. At the same time, you see that the oint is moving upwards so that its $y$-coördinate is also increasing. Hence, as $x$ increases, $y$ also increases.

(2) For every unit a point moves to the right and parallel o the $x$-axis, the point also moves upwards by one half unit. Thus an increase of 1 unit in $x$ produces an increase of $\frac{1}{2}$ unit n $y$.

(3) If a point moves left along the line until it has travell 4 units parallel to the $y$-axis, the $x$-coordinate of the point w have changed by 8 units. This is what takes place, for instan if we imagine a point to move along the line from (4, 0) $(-4, -4)$. It follows that a decrease of 4 in $y$ produces decrease of 8 in $x$.

(4) The point on the graph with $x$-coordinate 6 is (6, Therefore, when $x = 6$, $y = 1$.

(5) The point $(x, y)$ moves so that its $x$-coordinate l twice its $y$-coordinate equals 4.

### EXERCISES

What path is traced by the moving point $(x, y)$ as it mov in accordance with each of the following conditions?

1. $x = 3$.
2. $x = -4$.
3. $x = 0$.
4. $y = 8$.
5. $y = -5$.
6. $y = 0$.
7. $2x = 3$.
8. $3y = -4$.
9. $x + 8 =$
10. $y - 4 = 0$.
11. $x = y$.
12. $x = -y$.

Construct a table with three pairs of values for each of t following equations. Then draw the graph of the equati

13. $x + y = 8$.
14. $y - x = 5$.
15. $x - 2y = 0$.
16. $2x + y = 2$.
17. $x - 3y = -3$.
18. $3x - y = -6$.

19. Use the graph of Ex. 14 to find (*a*) the value of $y$ wh $x = 3$, (*b*) the value of $x$ when $y = -1$.

20. Use the graph of Ex. 15 to answer the following: What change occurs in $y$ as $x$ increases? (*b*) What change produced in $x$ by an increase of 1 in $y$?

21. Use the graph of Ex. 16 to answer the following: What change occurs in $x$ as $y$ increases? (*b*) What change produced in $y$ by an increase of 2 in $x$?

22. State in words the condition which the moving point ust satisfy as it traces the graph (*a*) in Ex. 15, (*b*) in Ex. 17.

## *Graph of a Linear System*

We can get some important information about linear stems of equations from graphs. Since each equation of the stem gives rise to a straight-line graph, it follows that the stem has for its graph two straight lines. In how many ays can two straight lines be related to each other? The nswer to this question will enable us to classify linear systems.

Let us examine the graphs of three different linear systems:

) . . . $x-y=5$ (2) . . . $x-y=5$ (3) . . . $x-y=5$
$2x+y=4.$ $x-y=-1.$ $2x-2y=10.$

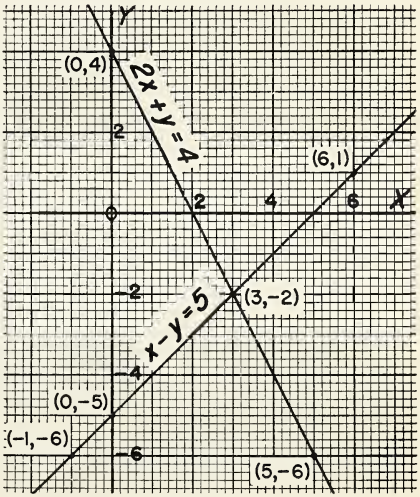
otting these equations in the usual way, we get the three llowing graphs.

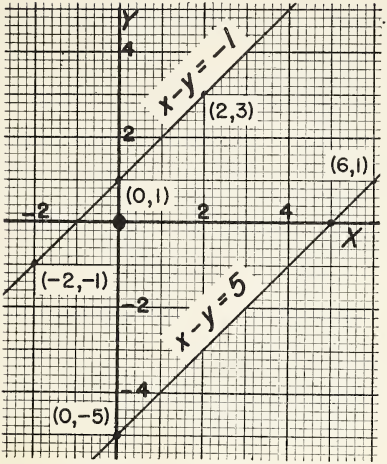
$x - y = 5$

| $x$ | $-1$ | $0$ | $6$ |
|---|---|---|---|
| $y$ | $-6$ | $-5$ | $1$ |

$2x + y = 4$

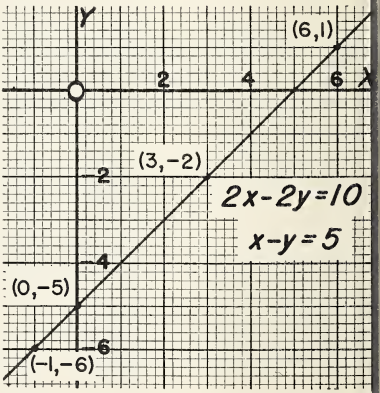
| $x$ | $0$ | $3$ | $5$ |
|---|---|---|---|
| $y$ | $4$ | $-2$ | $-6$ |






$x - y = -1$

| $x$ | $-2$ | $2$ | |
|---|---|---|---|
| $y$ | $-1$ | $3$ | |




$2x - 2y = 10$

| $x$ | $0$ | $3$ | $6$ |
|---|---|---|---|
| $y$ | $-5$ | $-2$ | $1$ |

Study these graphs carefully. Then try to describe thr types of linear systems that may arise. * * *

*System* 1. The graph shows that the two lines have exactly ne point in common. Therefore, there is only one number-air that satisfies both equations of (1). This means that ystem (1) has a ***unique*** solution.

*System* 2. The graphs have no point in common, and hence here is no number-pair that satisfies both equations of (2). ystem (2) has no solution; the two equations are ***inconsistent.***

*System* 3. The graphs coincide. Thus any point on the raph of the first equation is also on the graph of the second, nd its coordinates satisfy both equations. System (3) has an nlimited number of solutions; its two equations are equivalent.

***A system of two linear equations in two unknowns may have one solution, no solution, or an unlimited number of solutions.***

Since the graph of an equation may be regarded as the ocus of a point whose coordinates always satisfy the equation, e may think of the graphs of a linear system as a pair of such oci. The coordinates of all the points on one locus satisfy the rst equation and the coordinates of all the points on the other ocus satisfy the second equation. If the two loci have a oint in common, the coordinates of this point must satisfy *oth* equations. If the loci have no point in common, there xists no point whose coordinates satisfy both equations.

You should note that when the solution of a linear system is ound by plotting the two equations, your answer may be only pproximately correct. If the correct solution involves ractional values for $x$ and $y$, your graph paper often enables ou to make a good estimate and nothing better. When the raphs intersect on the corner of a square of the graph paper, e can find the solution accurately.

### EXERCISES

Draw graphs of the following systems of linear equatio
For each system, tell whether the equations have one soluti
no solution, or an unlimited number of solutions. Give uniq
solutions, where they exist.

1. $x + y = 2$
   $2x + y = 3.$

2. $x - y = 1$
   $x + y = 3.$

3. $x - y = 0$
   $x - 2y = 0.$

4. $x + y = 1$
   $y = -2.$

5. $x + 2y = 2$
   $x + 2y = 4.$

6. $2x - y = 4$
   $x = 3.$

7. $x + 3y = 3$
   $3x - y = 9.$

8. $2x + y = 4$
   $4x + 2y = 8.$

9. – 16. Verify the solutions found in Exs. 1-8 by solvi
the systems algebraically.

### REVIEW EXERCISES ON CHAPTER II

*A*

1. Draw the graph of the line $x - 4y = 0$.

2. Solve algebraically the system: $3x - 5y = -4$
   $x + 2y = -5.$

3. The sum of two numbers is 92 and their difference is 1
Find the numbers.

4. What is meant by (*a*) a variable, (*b*) the *locus* of
moving point?

5. Why is a first-degree equation in $x$ and $y$ called a *line*
*equation*?

6. A square with side 3 units is placed with one corner
the origin and two sides along the positive axes. Write t
coordinates of its four vertices.

7. Draw a graph of the linear system: $x = -y$
$x - y = 4.$

8. What locus is traced by the moving point $(x, y)$ as it oves in obedience to the law $x = 12$?

9. Draw the graph of the equation $4x - 3 = y$. Use the aph to find what change occurs in $y$ when $x$ increases. By w much does $y$ change when $x$ increases by 1 unit?

10. State in words the condition which the moving point , $y$) must satisfy as it traces the graph of $x = 5y - 4$.

11. How many solutions may a system of two linear equations $x$ and $y$ have?

12. If $m$ and $n$ are positive numbers, name the quadrant in hich each of the following points is found: $(m, n)$, $(m, -n)$, $-m, -n)$, $(-m, n)$. Where are $(-m, 0)$ and $(0, n)$ located?

13. What are the *coordinates* of a point?

14. Make a table of 10 pairs of values which satisfy the uation $x - 2y = 10$. From the table, find the change oduced in $y$ when $x$ increases by 2 units.

*B*

15. The difference of two numbers is 20. If 4 times the rger exceeds 7 times the smaller by 5, find the numbers.

16. Solve the following system and verify your answers:
$2x + 3y = 11$
$x - 2y = -5.$

17. Show by means of a graph that the following system has o solution:
$2x + y = 6$
$2x + y = 4.$

18. The difference of the terms of a proper fraction is 7. our times the numerator is 4 less than twice the denominator. ind the fraction.

Solve each of the following systems and verify your answe

**19.** $7m + 2n = 12$
$2m + 5n = -1.$

**20.** $\frac{1}{3}a - \frac{1}{2}b = 0$
$\frac{2}{3}a + \frac{3}{4}b = 7.$

**21.** $4.6x - y = 3.5$
$0.2x + 0.5y = -0.5.$

**22.** $3x - 4y = 12$
$x + 2y = 4.$

**23.** Solve the system of Ex. 22 graphically.

**24.** A total of \$800 is invested, some at 2% and some at 3
If the interest from both investments is \$19.50, how much
invested at each rate?

**25.** The perimeter of a rectangular pane of glass is 66 inch
If three times the length is 6 inches less than four times
width, find the dimensions of the pane.

**26.** A 96-foot length of chain is cut into two pieces so th
the larger piece is 6 feet longer than twice the smaller. H
long are the pieces?

**27.** Plot the graph of $x + 5y = 5$. Use the graph to answ
the following questions: (1) What happens to $y$ as $x$ increas
(2) What happens to $x$ as $y$ increases? (3) What change
produced in $y$ when $x$ increases by 5 units? (4) What con
tion must the point $(x, y)$ satisfy as it traces the graph?

**28.** Show by drawing a graph that there is a pair of valu
for $x$ and $y$ which satisfy the *three* following equations:

$$x + y = 7$$
$$2x - y = 5$$
$$x + 3y = 13.$$

**29.** Solve Ex. 10, page 30, using one unknown.

CHAPTER III

# FORMULAS AND GRAPHS

## *y Depends on x*

Recall the fact that the perimeter of a square depends on e length of its side. If the side has $s$ units of length and the rimeter $p$ units, the relation between the two variables is pressed by the *formula* $p = 4s$.

This formula furnishes one example of two variables which e so related that the value of one depends on that of the ner. Four other examples are given below. After each ample is given the formula which expresses precisely the ation between the variables.

(1) The area, $A$, of a circle depends on the length, $r$, of its dius.

$$A = \pi r^2$$

(2) The volume, $V$, of a cubical block depends on the gth, $e$, of an edge.

$$V = e^3$$

(3) The cost, $C$, of a number of articles at 10¢ each depends the number bought, $n$.

$$C = 10n$$

(4) The amount, $A$, of \$1 at 5% compound interest deds on the number, $n$ of interest periods.

$$A = 1.05^n$$

(5) If a rectangular field contains 10 acres, the number of ls in its width, $w$, depends on the number of rods in its gth, $l$,

$$w = 1600/l$$

Notice that in Example (1) when the length of the radius is known the area of the circle can be found; in (2) when the edge of the cube is known the volume can be found; in (3) when the number of articles is known the cost can be found; in (4) when the number of periods is known the amount can be found; in (5) when the length of the rectangle is known its width can be found.

*Whenever one variable y is related to another variable x in such a way that, when a value of x is given a value of y can be found, then y is called a function of x.*

This use of the word ***function*** corresponds to one use of the term in common speech. When, for example, we say that a boy's weight is a function of his height, we imply that if we knew a boy's height we would have some idea of his weight. Again, when it is said that a man's temper is a function of his digestion, it is implied that if the state of his digestion were known his temper could be predicted. Similarly, to say that the yield of grain is a function of the rainfall implies that if the rainfall in a given year were known one could estimate the yield of grain. In all these cases other factors than the one mentioned enter into the determination of the boy's weight or the man's temper or the yield of grain. This means only that the use of a term in common speech is less exact than its use in mathematics.

## Some Formulas and Their Meaning

Each of the functions mentioned on page 47 was expressed explicitly by means of a formula. Other formulas involving lengths, areas, and volumes were studied in Chapter I. We have already seen the usefulness of formulas as labour-saving devices. We shall now investigate their use in enabling us to tell how a change in one of the variables affects the other.

We shall consider, as examples, three of the above-mentioned formulas,

$$V = e^3, \qquad C = 10n, \qquad \text{and} \qquad w = 1600/l.$$

low do changes in $e$ affect $V$? It is clear that as $e$ increases $V$ lso increases, but much more rapidly. When the edge of a ıbe is doubled, its volume is multiplied by 8; when the edge trebled its volume is multiplied by 27, etc. The startling ıte of increase of $V$ is shown by the fact that, when $e$ increases om 1 to 100, $V$ increases from 1 to 1,000,000.

In $C = 10n$ a change of 1 in $n$ always causes a change of ) in $C$; doubling $n$ causes $C$ to double, increasing $n$ 5-fold ıuses $C$ to increase 5-fold, etc.

With $w = 1600/l$ the situation is different. Increasing makes $w$ decrease. If $l$ is doubled, $w$ is halved; if $l$ is multiied by 8, $w$ must be divided by 8; if $l$ is very large, $w$ is orrespondingly small.

**Changing the subject of a formula.** Each of the foregoing ›rmulas contains two variables. The formula expresses one these variables in terms of the other. It is possible to turn ıe formula around, as it were, and express the second variable terms of the first. Thus, from $C = 10n$ we get $n = C/10$. is the same relation which is expressed by these two ıuations; only the form of the relation has been altered. he first says that the number of cents spent is 10 times the ımber of articles bought; the second says that the number articles bought is 1/10 of the number of cents spent. In a milar way, from $V = e^3$ we get $e = \sqrt[3]{V}$ (read *the cube ›ot of* $V$). Also from $w = 1600/l$ we get $l = 1600/w$. his process of solving a formula for a variable which occurs it is sometimes called ***changing the subject of the ›rmula.***

## EXERCISES

1. In the formula $A = \pi r^2$, what is the effect on $A$ of ›ubling $r$?, of increasing $r$ 4-fold? What is the effect on $r$ diminishing $A$ to 1/9 of its former value? Express $r$ in :rms of $A$.

**2.** The formula $C = \frac{5}{9}(F - 32)$ expresses the relati between the temperature readings on a Fahrenheit and Centigrade thermometer. What increase in $C$ correspon to an increase of 180 in $F$? For what Fahrenheit temperatu does the Fahrenheit thermometer register above zero while t Centigrade thermometer registers below zero? Express $F$ terms of $C$.

**3.** In the formula $F = 100/r^2$, what is the effect on $F$ doubling $r$?, of halving $r$? What change in $r$ is necessary increase $F$ 25-fold? When $r$ increases from 1 to 10, what is t corresponding range of values of $F$? Express $r$ in terms of

**4.** The formula $V = lwh$ gives the volume of a rectangul block whose length, width, and height are $l$, $w$, and $h$ uni respectively. Express each of $l$, $w$, and $h$ in terms of the oth letters in the formula. If $l$ and $w$ are doubled, what chan in $h$ is necessary in order to leave $V$ unchanged? If $l$ is i creased by 50% and $w$ decreased by 50% while $h$ is unchang what is the percentage change in $V$?

**5.** The formula $d = rt$ gives the distance travelled in hours at the rate of $r$ units per hour. If $r$ is doubled ($a$) wh is the effect on $d$ if $t$ is fixed, ($b$) what is the effect on $t$ if $d$ fixed? If $d$ remains fixed, what percentage change in $t$ resu from ($c$) increasing $r$ by 100%, ($d$) decreasing $r$ by 50%?

## *Graphs of Formulas*

When only two variables are involved in a formula, t most striking way of showing the relation between them is means of a graph. In drawing graphs, as you recall fr Chapter II, we make use of pairs of numbers. How can build up a table of number-pairs from a formula? Yo previous work gives you the answer—by substitution. substituting numbers in the formula you build up a table values and from this table you construct a graph.

***Formula* ➡ *Table of Values* ➡ *Graph***

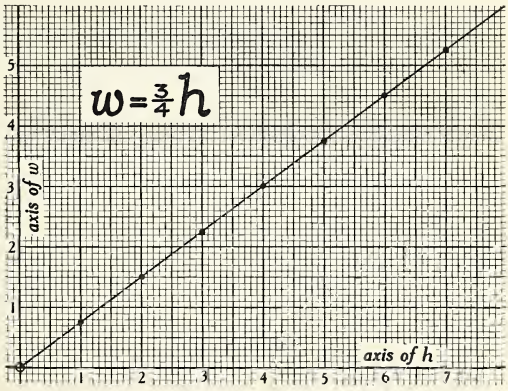
A simple example will best show the method. If a workman paid at the rate of 75¢ an hour, then for $h$ hours his wage, dollars, is

$$w = \tfrac{3}{4}h.$$

y giving to $h$ the values 0, 1, 2, 3, 4, etc., we calculate the rresponding values of $w$ and so build up the following table.

| $h$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | ... |
|---|---|---|---|---|---|---|---|---|---|---|
| $w$ | 0 | 0.75 | 1.5 | 2.25 | 3 | 3.75 | 4.5 | 5.25 | 6 | ... |

To represent the relation graphically we shall take the orizontal axis for the time axis or ***axis of h,*** and the vertical xis for the wage axis or ***axis of w.*** For each pair of values $h$ and $w$ in the table a point is to be plotted. If you use raph paper, which is marked off in squares, it will be very asy to locate the points. When you have plotted a number f points, it is apparent that they all lie on a straight line. his straight line is the graph of the formula $w = \frac{3}{4}h$.



Let us now look closely at this visual representation of formula. Even if we had no formula and no table, we co obtain much information from the graph. For example, increase of 2 in $h$, no matter where it is taken, will prod the same increase $3/2$ in $w$. An increase of 3 in $w$ alw results from the same increase 4 in $h$. Doubling $h$ will alw cause $w$ to double.

As a second example, consider the formula which gives area of a square. If the side of a square is $s$ units and area $A$ square units, we have

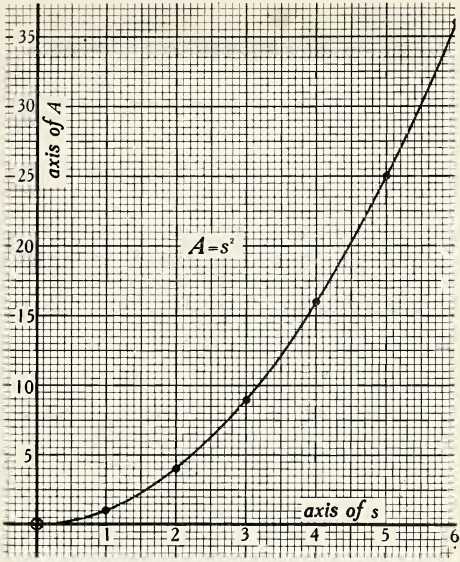
$$A = s^2.$$

First, we construct a table in which values of $A$ correspond to given values of $s$ are recorded.

| $s$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | . . |
|---|---|---|---|---|---|---|---|---|---|---|
| $A$ | 0 | 1 | 4 | 9 | 16 | 25 | 36 | 49 | 64 | . . |

Next, we take a sheet of squared paper, select an origin a axes and decide on the scales to be used on these axes. (Y had better take a look at how big the numbers are getti before deciding on your scale. Since $A$ eventually increa much faster than $s$, it will be well to choose a smaller unit the axis of $A$ than on the axis of $s$.) Now transfer the table the graph by plotting the points, one for each pair of valu of $s$ and $A$. The final step is to draw through these points smooth curve, making as close a fit as we can to the plott points. It is in this last step that you can train your ha and eye to produce a neat and workmanlike graph.

As this is not a straight graph it carries a somewhat differe message from that of the former one. As your eye follows t curve from left to right, you observe that it rises more and mo steeply. From $s = 0$ to $s = 1$, the rise is only 1 where from $s = 10$ to $s = 11$ the rise is 21. This means that




reases more and more rapidly with respect to $s$. The mula emphasizes this fact since, if $s$ is doubled, $A$ is multi- ed by 4.

As we take our leave of formulas and their graphs, let us nind ourselves again that we are here dealing with three ncies for presenting the same information. These are the mula, the table, and the graph. It may interest you to tice that, in the study of graphs, three aspects of your study mathematics are fused together. We may say that the mula gives the information algebraically, the table gives it ithmetically, and the graph gives it geometrically.

### EXERCISES

**1.** Draw a graph of the formula for the perimeter of an

equilateral triangle, $p = 3s$. (*a*) What is the character the graph? (*b*) What is the effect on $p$ of (i) increasing $s$ 4, (ii) increasing $s$ by $k$, (iii) multiplying $s$ by 3, (iv) multiply $s$ by $k$?

2. Draw the graph of the formula for the area of a ci $A = \pi r^2$. (Take $\pi = 3.1$). From the graph, describe a continuous increase in $r$ affects $A$.

3. A certain job requires 36 days' work of one man. men do the job in $t$ days, then the number of days' work is Hence $nt = 36$ and $n = 36/t$. Plot a graph of this funct From your graph read (*a*) how many men would do the in 8 days, (*b*) how many days 5 men would require.

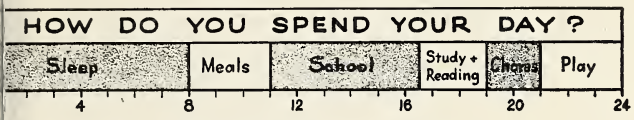
4. Draw a graph of the formula $C = \frac{5}{9}(F -$ What is the character of the graph? What temperat readings correspond to the points where the graph crosses axes?

## *Statistical Graphs. Three Types*

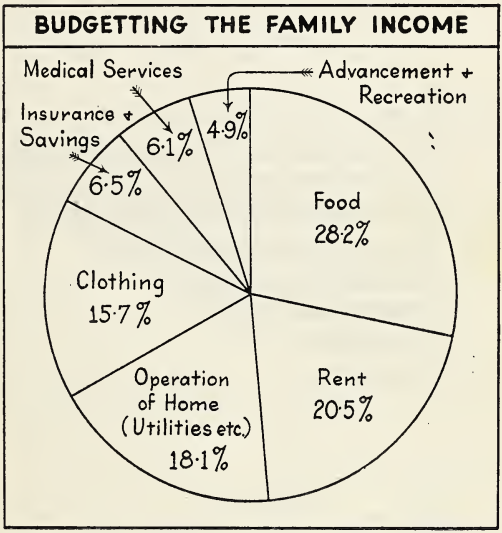
Many people have to prepare information regarding s things as population, manufactured goods, farm produ imports and exports, motor accidents, weather reports, so forth. Moreover, we all have to read and interpret so of this information. There are two ways in which such d can be concisely presented. One of these is by means of tabu ted lists of figures. The other is by means of graphs. You find in newspapers and magazines many examples of graphs different kinds ranging from pictorial charts or *pictographs* line graphs such as those we have already drawn.

Three types of graphs in common use will now be illustrat As you look at them decide what information each gr attempts to portray.

*A rectangular distribution graph*



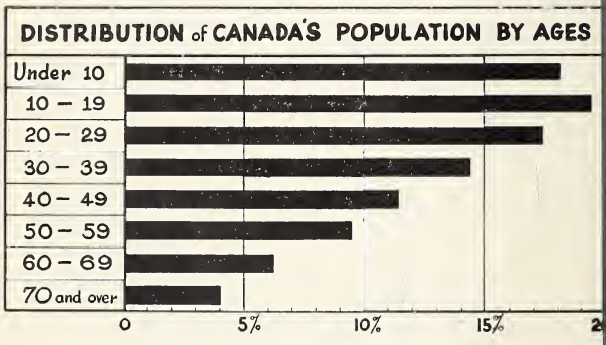

*A circular distribution graph*




In each of these two graphs, a certain whole (a 24-hour in the first, a family income in the second) is considered be made up of parts and the object is to show how much h part contributes to the whole. The rectangle and circle phs are particularly fitting for this purpose. In the rectan- graph the scale at the bottom corresponds to the 24 hours the day, so that the number of hours in each division is ly read off.

The construction of a circle graph requires a little m calculation. In the family-budget graph shown on page 55 read that 28.2% of the budget is spent for food. To repre this on the graph requires a sector which is 28.2% of the of the whole circle. The angle of the sector is therefore of 360°, which is approximately 102°. This angle is then lai with a protractor. The other angles are found similarly so the graph is constructed.

*A bar graph*




EXERCISES

1. From the bar graph shown above, state the percen of the Canadian population (*a*) between 40 and 50, (*b*) u 20, (*c*) 60 and over. (*d*) Are there more people in Can under 30 years of age or more over 30?

2. From the rectangular graph shown above, state many of the 24 hours are spent (*a*) in sleep, (*b*) at school in study and reading, (*d*) at play.

3. If a family income is spent according to the circle graph, te the fraction of it used for (*a*) rent, (*b*) insurance and ings, (*c*) clothing.

4. From the following data construct a bar graph to show lengths of the earth's five longest rivers.

iazon, 3800 miles; Congo, 3000 miles; Mississippi, 4200 miles; Nile, 3600 miles; Yangtze, 3400 miles.

5. **Where do school accidents happen?** The following ures are taken from the 1945 edition of *Accident Facts*, pub- ied in Chicago. Represent each set of figures on a rectang- r graph.

Accidents in school buildings: Gym., 36%; Hall and Stairs, 20%; Class Rooms, 15%; Shops, 13%; Other, 16%.

Accidents in school grounds: Unorganized activities, 45%; Football, 18%; Baseball, 11%; Other Organized Games, 16%; Apparatus, 10%.

6. Represent on a circle graph the distribution of the male pulation of Canada by occupations, as follows:

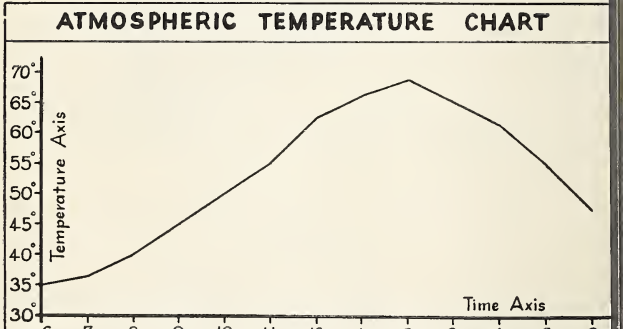
Agriculture, 31.7%; Manufacturing, 17.1%; Trade, 8.1%; Transportation, 7.6%; Fishing, Trapping and Logging, 3.9%; Mining, 2.1%; Construction, 6.0%; Service, 9.4%; Clerical, 5.4%; Others, 8.7%.

## *Line Graphs of Statistical Data*

Whenever two variables $x$ and $y$ are related, that is when a function of $x$, the relation between them can be exhibited means of a graph. Of all the types of graphs, those which easiest to construct and which at the same time give the ormation in a very usable form are the line graphs. These re the graphs used in Chapter II and illustrated on pages 51 d 53.

Suppose that records are kept of the atmospheric tempera- e during the 24 hours of a day. Then the temperature is a

function of the time. In drawing its graph we use the hori tal axis as the ***time axis.*** On the graph below, this axi marked off in spaces each of which represents one hour. origin is marked 6 a.m. and the point marked 12 repres noon. The vertical axis is the ***temperature axis.*** scale on it shows degrees Fahrenheit which are marked intervals of 5°, beginning with 30° at the origin. Now supp we wish to put on the graph the fact that the temperatur 9 a.m. is 44°. From the point marked 9 on the time axis measure upward a distance corresponding to 14°. Since vertical interval is 5°, we require $2\frac{4}{5}$ of these intervals. this way we locate a point on our graph. Having found cessive points corresponding to the hours, we draw thro them a line. This line may be a broken line consisting several straight portions or it may be "smoothed off" so a obtain a smooth curve. The rise and fall of this curve sh clearly the rise and fall of the temperature during the day




## EXERCISES

Construct broken-line graphs, properly labelled, to rep sent the data in 1 and 2.

1. The National Debt of Canada, in millions of dollars:

1871, 78; 1881, 155; 1891, 238; 1901, 268;
1911, 340; 1921, 2341; 1931, 2262; 1941, 3649.

2. The production of Douglas Fir, in Canada, in millions feet, board measure.

1927, 1437; 1928, 1517; 1929, 1605; 1930, 1255;
1931, 936; 1932, 642; 1933, 818; 1934, 1032;
1935, 1115; 1936, 1422; 1937, 1407; 1938, 1402;
1939, 1595; 1940, 1692; 1941, 1525; 1942, 1540;
1943, 1234.

3. Your graph of Ex. 1 should recall to you some facts of nadian history and provide the answers to the following estions:

(*a*) Which was the decade of railway building?

(*b*) In which decade did a Great War occur?

(*c*) In which decade did the government borrow the least money? What conclusion do you draw regarding the condition of the country in this decade?

(*d*) What is likely to be the character of the graph in the decade after 1941?

4. From your graph of Ex. 2, answer the following questions:

(*a*) Describe how the production of Douglas Fir has varied from 1927 to 1943.

(*b*) How do you account for the depression in the curve?

(*c*) For what year is the graph rising most steeply? What does this mean?

(*d*) For what year is the graph falling most steeply? What is the explanation?

## REVIEW EXERCISES

*A*

**1. Canadian immigration.** Represent by a bar gra Canada's immigration in the year 1937-38: from the Uni States, 5653; from Central Europe, 4773; from the Uni Kingdom, 3351; from other countries, 1878.

**2. Canal traffic.** Represent on a bar graph the follow comparison of three canals, by net tons of traffic (1937).

Sault Ste. Marie, 87,637,000;
Suez, 36,129,000;
Panama, 28,108,000.

**3. Where is merchandise sold?** Represent on a rect gular graph the following distribution of sales by retail sto of different types: Food stores, 25.2%; General merchand stores, 19.1%; Automotive sales, 17.5%; All other sto 38.2%.

**4.** (*a*) Draw a graph of the formula for the perimeter o square, $p = 4s$. (*b*) What is the character of the gra (*c*) Two squares have sides of 10 inches and 100 inches spectively. What is the effect on the perimeters of (i) doubl all the sides, (ii) increasing all the sides by 2 inches?

**5. Canada's population.** (*a*) Make a bar graph to sh the growth in population of Canada according to the las censuses.

1891, 4,833,239; 1901, 5,371,315; 1911, 7,206,643
1921, 8,787,949; 1931, 10,376,786; 1941, 11,506,65

(*b*) From your graph, tell in which decade the growth population was most rapid.

**6. Passenger cars in Canada.** (*a*) Represent by a graph the number of passenger cars in Canada in the per 1905 to 1940.

1905, 553; 1910, 5890; 1915, 60,688;
1920, 251,945; 1925, 641,186; 1930, 1,055,514;
1935, 989,744; 1940, 1,234,637.

Give an explanation for the depression in your curve.

**7. High buildings.** Represent by a bar graph the heights the following buildings:

| | |
|---|---|
| Great Pyramid, | 480 feet; |
| Eiffel Tower, | 984 feet; |
| Empire State Building, | 1250 feet; |
| Chrysler Building, | 1046 feet; |
| R. C. A. Building, Rockefeller Centre, | 850 feet; |
| Woolworth Building, | 792 feet. |

*B*

**8. Three industries in war time.** (*a*) Represent the lowing information by means of bar graphs, one for each lustry.

Number of persons employed in three industries of Canada.

| | Aircraft | Farm Implements | Shipbuilding |
|---|---|---|---|
| 1938.... | 1617 | 6458 | 3596 |
| 1939.... | 3596 | 5306 | 3491 |
| 1940.... | 10,348 | 6857 | 9707 |
| 1941.... | 26,661 | 8804 | 21,240 |
| 1942.... | 44,886 | 11,853 | 50,132 |

In which of the three graphs do the bars lengthen most wly? What is the explanation?

**9. How far away are the planets?** Representing the sun a point, draw lines radiating from it to show the distances of planets. The distance of the earth from the sun is taken 1.

Mercury, 0.39; Venus, 0.72; Earth 1.00;
Mars, 1.52; Jupiter, 5.20; Saturn, 9.54;
Uranus, 19.18; Neptune, 30.05; Pluto, 39.51.

**10. The circumference of a circle.** Draw a graph of formula for the circumference of a circle, $C = 2\pi r$, tak $\pi = 22/7$. What is the character of the graph? Comp it with the graph of the perimeter of a square (Ex. 4).

**11. Safety first!** The number of accidental deaths Canada each year reaches tragic proportions. In 1943 total number was 7385, distributed according to the follow figures. Represent this information on a circle graph. Fa 23%; Motor Vehicle, 19%; Drowning, 14%; Burns, 11 Railroad, 4%; Poison Gases, 2%; Other, 27%.

**12. Income Tax.** An increasingly important source Dominion revenue has been the tax on incomes. In following table the number after each year gives the inco tax in millions of dollars.

1936, 83; 1937, 102; 1938, 120; 1939, 142; 1940, 134; 1941, 248; 1942, 510; 1943, 910; 1944, 1152.

(*a*) Represent this information by a broken-line graph. Between what years is the graph steepest? What is explanation?

**13. We go to school.** Out of each 100,000 pupils Canada in Grade I the numbers of those who continue to higher grades are as follows:

Grade I, 100,000; Grade II, 99,284; Grade III, 98,2
Grade IV. 96,112; Grade V, 92,237; Grade VI, 85,9
Grade VII, 77,163; Grade VIII, 66,829; Grade IX, 48,2
Grade X, 33,329; Grade XI, 20,191.

Represent this information by a line graph.

**14. Where do apples come from?** The average exp of apples by different exporting countries for the 5-year peri 1928 to 1932, are listed in percentages below. Show this formation on a circle graph.

United States, 57.4%; Canada, 15.2%; Australia, 10.9% Italy, 5.7%; Yugoslavia, 5.1%; New Zealand, 3.5 Others, 2.2%.

**15. The Girl Guides.** The total number of members of e organization of Girl Guides in Canada, over a 10-year riod, is given by the following figures. Round off these mbers to 2 figures and represent them on a line graph.

36, 48,392; 1937, 63,308; 1938, 57,266; 1939, 59,206; 40, 61,888; 1941, 67,531; 1942, 58,612; 1943, 60,160; 44, 64,166; 1945, 64,236.

**16. The falling-body formula.** If a body, starting from t, falls $s$ feet in $t$ seconds, then $s = 16\,t^2$, very nearly. aw a graph of this formula, being careful to select scales your two axes so as to obtain a convenient curve.

*C*

**17.** A. Walker walks from his home, Spruce Nook, to gleton, 8 miles away, at the rate of 4 miles an hour. He nains at Eagleton for 2 hours and then returns home at 2 les an hour. Construct a line graph showing Mr. Walker's tance from Spruce Nook from the time he left until he urned. Use a horizontal *time* axis and a vertical *distance* s.

**18.** Draw the graph of the formula

$$d = 0.045\,r^2 + 1.1\,r,$$

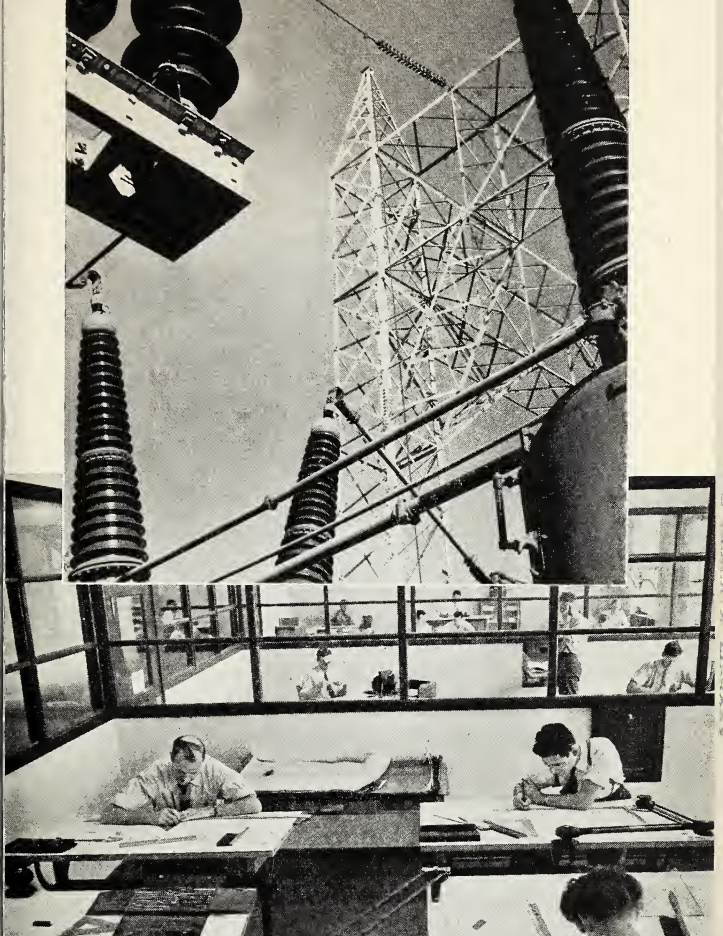
ich gives the stopping distance of a car at different speeds.

**19.** The Jewel restaurant serves 60¢ meals. If the average t of a meal is 30¢ and the daily cost of labour and rent is 2, then the daily profit for $n$ meals served is $p$ cents, where

$$p = 30\,n - 1200.$$

oosing a convenient scale for each axis, draw a line graph to w the relation between $n$ and $p$. What is the interpretation of the fact that a part of the graph is below the axis of $n$?

**20.** The directors of Camp Almac planned trips for t groups of boys. Wilbur's group started from Almac, travel $1\frac{1}{2}$ miles east and then turned northeast. John's gr started from Bilnuk, 5 miles east of Almac, travelled 1 m south and then turned northwest. They were told tha party would meet them in the bush with their supper. F graphically the point at which the provisioning party sho place itself in order to intercept both Wilbur's and Joh groups.



*The rigid property of triangles is a factor in the strength of steel framework.*
*(Courtesy of Director of Public Information, Ottawa)*

*m: Geometric drawing is of great importance in engineering research.*
*(Courtesy of The National Film Board)*

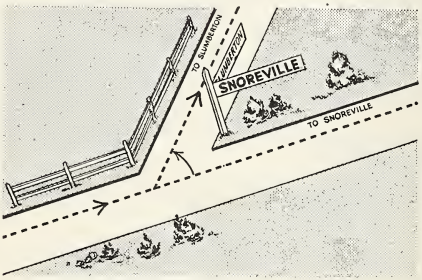
CHAPTER IV

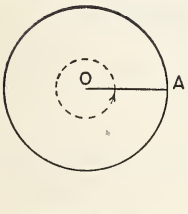
# THE ROTATION OF A STRAIGHT LINE

*"There is no royal road to geometry."*

Menaechmus

Suppose that you are going by bicycle to the village Slumberton. When you come to the intersection of Slumberton and Snoreville roads you must change directi The *amount of turning* made by your handlebars is an ***angle***

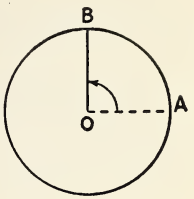


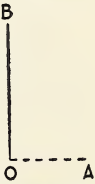

**The complete revolution.** The ill tration shows one door of a set of swing doors. Let us suppose that it was swu completely around once. In that complete revolution, the door turn through an angle of three hundred sixty degrees (360°).

**A quarter of a complete revolution.** If the door mad quarter of a revolution, so that it came to rest in the posit

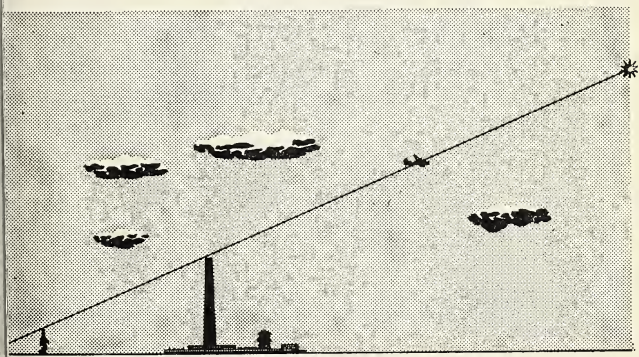
ow shown, the angle through which it rned was a quarter of 360° or 90°. his square angle is called a ***right angle.*** otice a few of the right angles in the om in which you are sitting.




The door turned from the position *OA* the position *OB*. These two lines *A* and *OB*) are called the ***arms*** of the gle. You should know that the length of the arms has nothing to do with the size of the angle. Rotate one leg of a small pair of dividers through a quarter revolution. Also, rotate one leg of a pair of blackboard compasses through a quarter revolution. The two angles are equal.



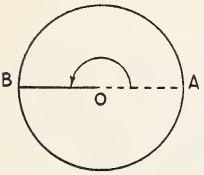

Furthermore, the two arms of the angle need t be of the same length. You can imagine at the line *OA* rotated until it lay along *OB*.



The angle has the same size irrespective of whether the arm reaches to the man, the chimney, the aircraft or the star.

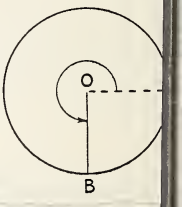
These two facts, namely, that increasing or diminishi
the arms has no effect on the angle and that the arms may
unequal, are true of all angles.

**The straight angle.** This time we shall turn the door un
it is exactly in line with its original position. The ang
formed is often called a ***straight ang***
Obviously, to complete the revolutio
it would be necessary to turn the do
through another straight angle.

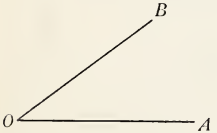



Two straight angles are equal to 36
Therefore, one straight angle is 18
How many right angles are there in o
straight angle?

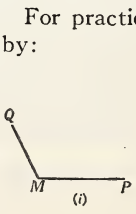
**The angle of 270°.** When the door has made three-quarters of a complete revolution, it is in the position shown. The angle turned through is three-quarters of 360°, that is to say, 270°. How many right angles are there in three-quarters of a complete revolution?

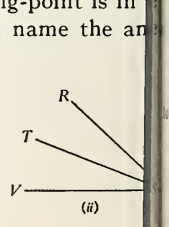



**Naming angles.** To refer conveniently to an angle,
letter the arms. In the illustration
is the point at which the lines $AO$ a
$BO$ intersect. The angle is cal
$AOB$ or $BOA$. Note that in both cas
the letter at the turning-point is in
middle. For practice, name the an
formed by:

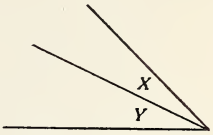



(*a*) $MP$ and $MQ$;
(*b*) $SR$ and $ST$;
(*c*) $ST$ and $SV$;
(*d*) $SR$ and $SV$.

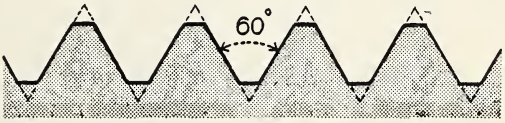


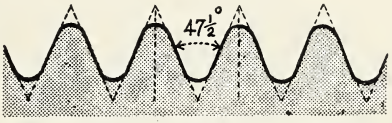



Frequently, we use a single letter to denote an angle. Thus (i) above, we may call the angle ther $PMQ$ or $M$. In the accompany- g figure, two angles are designated by e single letters $X$ and $Y$. In the ;ure above, why would it be objec- ɔnable to refer to the angle $TSV$ as e angle $S$?




Instead of writing the word *angle*, we frequently use the mbol $\angle$. Thus, instead of writing *angle* $VST$, we may ite more briefly $\angle VST$.




The angles of gear teeth

## EXERCISES

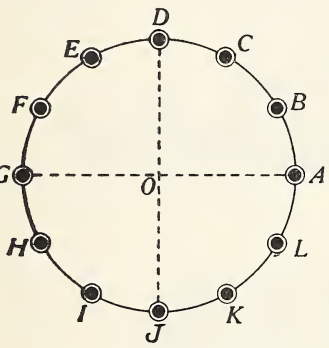
1. Through what angle do you turn in making (*a*) a left rn, (*b*) a right incline, (*c*) an about turn?

2. State the length of time required
   (*a*) for the minute hand of a clock to turn through 90°;
   (*b*) for the hour hand of a clock to turn through 90°.

3. State the number of degrees through which a spoke of wheel would rotate when making each of the following turn

(*a*) two revolutions;
(*b*) one-sixth of a revolution;
(*c*) one-eighth of a revolution;
(*d*) one-twelfth of a revolution;
(*e*) one-three hundred and sixtieth of a revolution.

4. The earth rotates on its axis once each day. Throu how many degrees does it rotate per hour?




5. *A*, *B*, *C*, etc. are rive evenly placed in a circul metal patch. Calculate t number of degrees in each the following angles:

(*a*) *AOB*; (*e*) *AOF*;
(*b*) *AOG*; (*f*) *BOC*;
(*c*) *AOE*; (*g*) *COE*;
(*d*) *AOC*; (*h*) *BOF*.

6. What fraction of revolution is each of t following angles?

(*a*) 120°; (*d*) 45°; (*g*) 108°; (*j*) 240°;
(*b*) 15°; (*e*) 60°; (*h*) 135°; (*k*) 270°;
(*c*) $22\frac{1}{2}$°; (*f*) 72°; (*i*) 225°; (*l*) 300°.

7. State the number of degrees turned through from 3 p. to 4 p.m. on the same day by (*a*) the hour hand of a clock, the minute hand.

**The magnetic compass.** Examine a magnetic compa You should be familiar with the following directions:

(*a*) North, South, East and West;
(*b*) North-East (NE); NW; SE and SW.

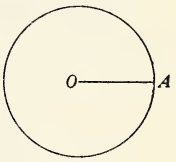
**8. State the angle between the directions:**

(*a*) N and E; (*d*) S and SE; (*g*) NE and SE;
(*b*) E and W; (*e*) W and SW; (*h*) NE and SW;
(*c*) N and NE; (*f*) W and NW; (*i*) N and SW.

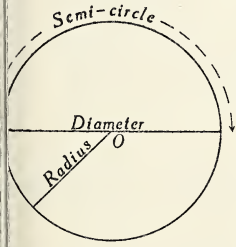
**A familiar figure.** Take any point on a sheet of paper. nagine that a second point must obey the following rules: ) it must be on the paper; (*b*) it must be neither more nor ss than two inches from the first point. Mark a number of ssible positions of the second point. Along what path must lie?

* * *

As you have seen, the path traced out a point moving on a level or ***plane*** surface so that it is always at a fixed distance om a fixed point is a ***circle***. The plane rface need not be horizontal, but may be ted at any angle to the horizontal.




The fixed point *O* is called the ***centre*** of the circle. Any line drawn from the centre to the circle is called a ***radius*** (plural, ***radii***). A straight line which passes through the centre and which has its ends on the circle is called a *diameter*. How does a diameter compare in length with a radius?




The end points of any diameter divide the circle into two equal parts each of which is a ***semicircle.***

EXERCISE

A pupil who was asked to define a circle said, "A circle is e path of a movable point which is always a fixed distance om a fixed point." Criticize his definition.

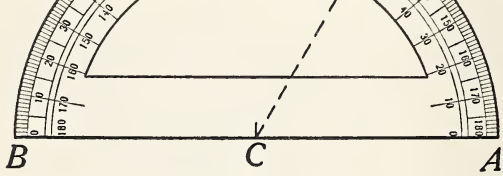
**How to measure an angle.** In your box of instrument you will find a *protractor* similar to the one shown in th illustration on this page. Examine it closely.

How would you describe the curved line from $A$ to $B$?

At the middle of the straight portion is a mark. Th point $C$, where this mark meets the lower edge, is the cent of the circle of which $APB$ is half.

We have drawn $CP$, one of the radii of the circle. Imagi that $CP$ is movable and hinged at $C$. If $CP$ set out from th position $CA$ and turned into its present position $CP$, then th angle formed ($ACP$) is, as you can see, 60°.

Perhaps you are wondering about the 120° mark at th same place.

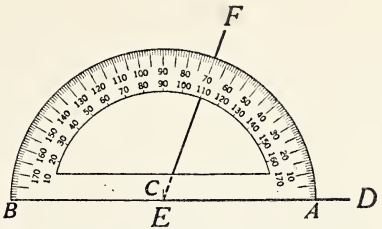



If $CP$ had set out from $CB$ and had turned in the *clockwi* sense—that is, the sense in which the hands of a clock move it would have made $\angle BCP$, which is 120°.

Clearly, $ACP$ is less than 90°. Use your common sen and you won't have any trouble in deciding which reading take.

* * *

To measure $\angle DEF$, place the protractor so that $C$ exactly at $E$ and so that $CA$ is placed accurately along $ED$.

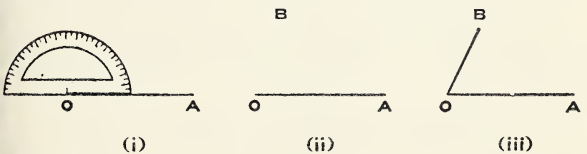



Now note where $EF$ meets the semicircle. Which set of aduations must be used? How many degrees are in $\angle DEF$?

> ***The accuracy with which an angle is measured depends to a large extent on the care with which the protractor is placed.***

**Making an angle of a given size.** Draw a line $OA$. From draw a line $OB$ so that $\angle AOB$ is 65°. Do this easy exer-e without assistance. Then read the following lines and d out if you have carefully followed the procedure described.

You are given a straight line $OA$, and are required to draw ine $OB$ so that the $\angle AOB$ is 65°.




(i) (ii) (iii)

Place your protractor accurately along $OA$ so that the ark on the straight edge is exactly at $O$ (Fig. i). Using sharp pencil, make a point at the 65° mark (Fig. ii). Join s point to $O$ (Fig. iii).

## EXERCISES

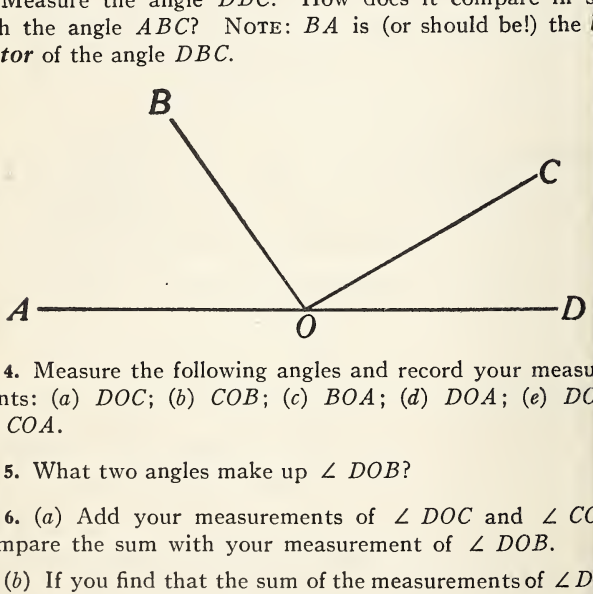
**1.** Construct the angle which the hands of a clock make (*a*) 2 o'clock, (*b*) 3 o'clock, (*c*) 6 o'clock.

**2.** Construct the following angles: (*a*) 185°; (*b*) 225°; ( 270°; (*d*) 315°.

**3.** Draw any angle $ABC$. On the side of $BA$ remote fr $BC$, draw a line $BD$ so that the angle $ABD$ is equal to t angle $ABC$. (Use your protractor.)

*Hint*: Draw a rough freehand sketch. Then plan yo procedure.

Measure the angle $DBC$. How does it compare in si with the angle $ABC$? NOTE: $BA$ is (or should be!) the *b* ***sector*** of the angle $DBC$.




**4.** Measure the following angles and record your measur ments: (*a*) $DOC$; (*b*) $COB$; (*c*) $BOA$; (*d*) $DOA$; (*e*) $DO$ (*f*) $COA$.

**5.** What two angles make up $\angle DOB$?

**6.** (*a*) Add your measurements of $\angle DOC$ and $\angle CO$ Compare the sum with your measurement of $\angle DOB$.

(*b*) If you find that the sum of the measurements of $\angle D$ and $\angle COB$ is equal to the measurement of $\angle DOB$, ha

ou proved conclusively that your work is accurate? Give our reasons.

(*c*) If you find that the two results are not equal, have ou proved that your work is inaccurate? Why?

7. Compare the sum of the measurements of $\angle COB$ d $\angle BOA$ with the measurement of $\angle COA$.

8. Compare the sum of the measurements of $\angle$s$DOC$, $OB$ and $BOA$ with the measurement of $\angle DOA$.

9. Write the following statements in your work-books and l in the blanks:

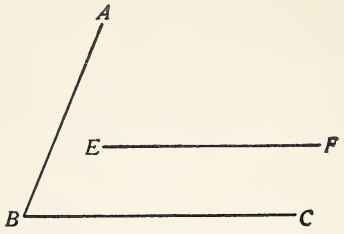
(*a*) $\angle BOD - \angle BOC = \angle$ ;
(*b*) $\angle AOC - \angle = \angle AOB$;
(*c*) $\angle AOD - \angle = \angle AOB + \angle COD$.

**Estimation of the size of an angle.** The ability to make a ose estimate of the size of an angle is sometimes useful. You ll now be given practice in estimating the number of degrees angles drawn on paper. Later, you will have an opportunity of estimating angles out-of-doors.

Draw an angle and estimate its size. Then measure it refully. Record your estimate, the actual size, and the ror of estimate in a table similar to the following:

| *Estimated size of angle* | *Actual size of angle* | *Error of estimate* |
|---|---|---|
| 20° ..... | 13° ..... | + 7° |
| 35° ..... | 45° ..... | − 10° |
| 60° ..... | 55° ..... | + 5° |

Repeat estimation and measurement until you become asonably expert in the former.




**Copying an angle.** A man who was laying out walks in h garden wished to obtain a line $DE$ so that $\angle DEF$ is equal t $\angle ABC$.

We shall try to solve this little problem on paper. The we shall see how it could be solved in the garden. Draw th lines in your work-book. Then, using your protractor, obtai the line $DE$.

* * *

We have mentioned that the Greeks regarded geometry as game. One rule of the game was that the use of a protract was forbidden. Another forbade the use of a ruler *wi graduations on it.* It was permitted to draw straight lin with the aid of an ungraduated ruler or straight edge. Ther was nothing unsportsmanlike in using the compasses.

You have copied the angle by using your protractor. Yo could also have copied it by using tissue paper. (This woul have given Euclid a fit.) Now play under the old rules: *Usi straight edge and compasses only, make an angle DEF equal the angle ABC.*

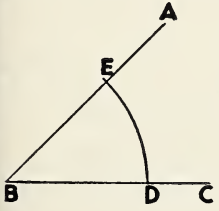
* * *

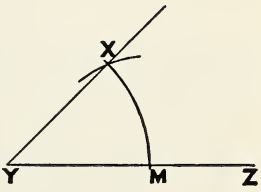
## A SUGGESTION

***Some run for help whenever they meet a difficulty. Others, more independent, scorn to seek assistance before they exert to the utmost their own powers. The effort you make in trying to solve a problem is often worth more than the obtaining of the answer.***

***The construction of the required angle is given below. Compare your method with the one described. If your construction is different from the one described, bring it to your teacher's notice. Correct solutions differing from those in the textbook interest the teacher.***

You are given the $\angle B$ and the straight line $YZ$, and are quired to draw a straight line $XY$ so that $\angle Y = \angle B$.




Open your compasses any distance $BD$. With $B$ as centre, aw a circle. (You will find that you need only part of the cle.) Part of a circle is called an ***arc.*** Let this arc cut $A$ at $E$.

With centre $Y$ and the same radius, draw an arc cutti
$YZ$ at $M$.

Place your compass point at $D$ and adjust the instrume
until your pencil point is at $E$. With centre $M$ and radi
$DE$, draw an arc. Let this arc intersect the other one at
Join $XY$.

Measure the angles $B$ and $Y$. Are they equal?

* * *

This operation can be performed out-of-doors by using
rope and pegs instead of compasses.

**Those set squares.** Place one of the set squares on yo
paper. Draw lines along the edges. Remove the set squa
and produce the lines to meet. Measure the three angl
formed. Similarly, measure the three angles of the other s
square.

One of the set squares provides you with a 'ready mad
angle of 45°. The other gives you the angles 30° and 6
Both set squares are useful for the construction of right angle

* * *

DEFINITION: ***If the angle between two straight lin
is a right angle, each line is said to be perpendicular
the other.***

### EXERCISES

**1.** Give a description of an angle.

**2.** How many degrees are there in one revolution?

**3.** What is (*a*) a right angle, (*b*) a straight angle?

**4.** How would you crease a piece of paper so as to form
right angle?

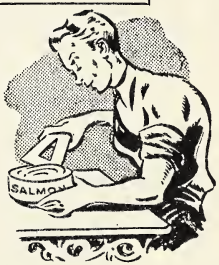
**5.** (*a*) When are two straight lines said to be perpendicul
to each other? Illustrate your answer by a drawing.

(*b*) In your room find examples of (i) two perpendicul

aight lines, (ii) three lines each perpendicular to the others, i) three lines each perpendicular to another line.

**HINT**

***Keep all your instruments from injury. A pair of compasses with a blunt point is useless. Bent metal set squares are merely scrap.***




6. Draw two straight lines making an angle of 270°.

7. Draw two angles, one less than 90°, the other greater an 90°, but less than 180°. Measure the angles.

8. Draw an angle. Measure it with your protractor. ing compasses and straight edge, construct an angle double size. Check the accuracy of your result by careful measure- nt.

9. How would you make an angle equal to a given angle (*a*) ing the protractor, (*b*) using compasses and straight edge ly? Make drawings to illustrate your answers.

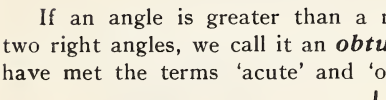
10. Draw a straight line $AB$. Take any point $Q$ in $AB$ d any point $P$ not in $AB$. Join $PQ$. Using straight edge d compasses, draw a line $PR$ so that $\angle RPQ = \angle PQB$.

11. Give a description of each of the following: (*a*) a circ
(*b*) a radius, (*c*) a diameter.

## *Acute, Obtuse and Reflex Angles*

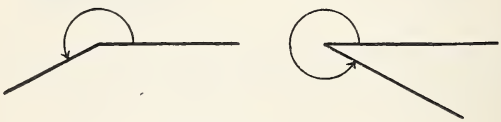
An ***acute ang***
is one that is le
than a right angle.

If an angle is greater than a right angle, but less th
two right angles, we call it an ***obtuse angle.*** No doubt y
have met the terms 'acute' and 'obtuse' already. A shar



witted person is som
times described
acute, a dull-witted pe
son, as obtuse.

A ***reflex angle*** is greater than a straight angle but is le
than two straight angles.

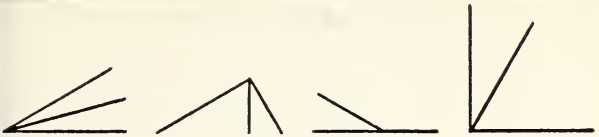


Above are two examples of reflex angles.

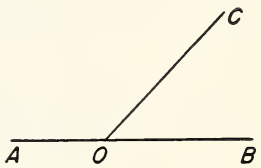
### EXERCISE

What name would you apply to each of the following angle
(*a*) 37°, (*b*) 225°, (*c*) 180°, (*d*) 135°, (*e*) 360°, (*f*) 181°, (*g*) 17
(*h*) 359°, (*i*) 91°, (*j*) 89°.

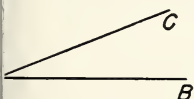
**Adjacent angles.** When two angles with a common vert
have a common arm between the other two arms, they are call
***adjacent angles.*** Here are four pairs of adjacent angles:



**Supplementary and complementary angles.** If the sum of two angles is 180°, the angles are said to be ***supplementary***. For instance, 80° and 100° are supplementary angles. 60° is the supplement of 120°; 120° is the supplement of 60°.




If $AO$ and $OB$ are in the same straight line, then the angles $BOC$ and $COA$ are supplementary.




If the sum of two angles is 90°, the angles are said to be ***complementary***. Thus, 30° is the complement of 60°; 60° is the complement of 30°. In the figure, $\angle AOB = 90°$. Hence, according to our definition, the angles $BOC$ and $COA$ are complementary. It will be noted that $\angle BOC$ and $\angle COA$ are complementary adjacent angles.

### EXERCISES

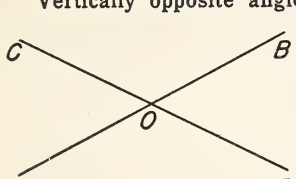
1. What is the supplement of (*a*) 50°, (*b*) 130°, (*c*) 78°, (*d*) 90°?

2. Can two obtuse angles be supplementary? Explain.

3. Give the complement of (*a*) 20°, (*b*) 70°, (*c*) 30°, (*d*) 45°.

4. One angle is double another and the two are compl mentary. What is the size of each angle?

**Vertically opposite angles.** In the accompanying figur the straight lines $AB$ and $C$ intersect at $O$. The angl $AOD$ and $BOC$ are said to ***vertically opposite angle*** $AOC$ and $DOB$ form anoth pair of vertically opposi angles.




## AN IMPORTANT CLASS EXERCISE

Draw an angle $AOB$ which is less than 180°. Measu $\angle AOB$ and subtract the measurement from 180°. At $O$, the straight line $OB$, and on the opposite side from $OA$, make angle $BOC$ equal to the difference. Produce $AO$ to $X$.

What do you observe about the three straight lines, $AO$ $AO$ and $OC$? Compare your findings with those of oth pupils.

It will be noted that (*a*) $\angle$s $AOB$ and $BOC$ are adjace angles, (*b*) $\angle$s $AOB$ and $BOC$ are supplementary angles, the arms $OA$ and $OC$ of these supplementary adjacent angl are in the same straight line.

The conclusion derived from this class exercise is an i portant one. Although its truth is not established by maki constructions, you are assured that satisfactory proof can given. Let us re-state the conclusion:—

*If two adjacent angles together form two right angles, th exterior arms are in the same straight line.*

## EXERCISES ON CHAPTER IV

*A*

1. Draw a line $AB$ 2 inches in length. At $A$, make $\angle BAX$ ual to 60°. At $B$, make $\angle ABY$ equal to 30°. Let $AX$ d $BY$ intersect at $C$. Measure $\angle ACB$.

2. At the point $O$ in the straight line $AB$, $OC$ is drawn so at $\angle BOC$ is acute. Measure $\angle$s $BOC$ and $COA$. What is e sum of the two angles? How would you describe the two gles?

3. With the aid of a figure, explain what is meant by flex angle'.

4. Make a sketch of each of the following:

(*a*) an acute angle; (*b*) an obtuse angle; (*c*) two adjacent gles; (*d*) two adjacent complementary angles; (*e*) two jacent supplementary angles; (*f*) vertically opposite angles.

5. Give the supplement of (*a*) 41°, (*b*) 150°, (*c*) 27°, (*d*) 1°, 179°, (*f*) 90°, (*g*) $n°$, (*h*) $(180-n)°$, (*i*) $A + B$.

6. Give the complement of (*a*) 25°, (*b*) 60°, (*c*) 1°, (*d*) 10°, $n°$, (*f*) $(90-n)°$, (*g*) $(m + n)°$, (*h*) $(180-n)°$.

7. Is the complement of an angle necessarily acute? :plain.

8. (*a*) If you double the size of an acute angle, do you cessarily obtain an obtuse angle?

(*b*) Is the half of an obtuse angle necessarily acute?

9. If the complements of the angles $A$ and $B$ are equal, iat conclusion can be drawn?

10. If the supplements of $A$ and $B$ are equal, what con- usion can be drawn?

11. If two angles are supplementary, is the one necessarily ute and the other obtuse?

*B*

**12.** If the numbers of degrees in two complementary angl are $2x$ and $3x$, find $x$.

**13.** If an angle and half the angle are supplementary, wh is the size of the angle?

**14.** Two angles are supplementary and one is three tim the other. What is the size of each?

**15.** The difference of two complementary angles is 2 What are the angles?

**16.** The difference of two supplementary angles is 2 What are the angles?

**17.** If the numbers of degrees in two supplementary angl are $2x-50$ and $4x-10$, find the two angles.

**18.** The supplement of an angle is three times its compl ment. What is the angle?

**19.** Show that the supplement of the complement of a angle is greater than the angle by one right angle.

**20.** Show that the complement of the supplement of a angle is less than the angle by one right angle.

CHAPTER V

# JRTHER EXPLORATIONS IN GEOMETRY

*"Why is this thus? What is the reason of this thusness?"*
Artemus Ward

With the help of your instruments, you have already ined a considerable store of geometrical information, and, the very same way, you could make extensive further ogress. We shall, however, make a new approach to the oject, retaining the straight edge and compasses, but dis-rding the protractor, set squares and ruler. Occasionally, e latter instruments will be used in exercises to prevent you m forgetting all about them.

Don't imagine that some instruments are sacred and that ers are beneath contempt. We are following old rules of e game, rules laid down by the ancient Greek mathematicians, discarding all but the straight edge and compasses. How-er, astronomers, engineers and many others who make use geometry, have no hesitation in employing other instruments.

You may ask the fair question: "Why are the geometrical easuring instruments to be turned out into the cold in edience to some old rule when it is admitted that they can of service in the gaining of new knowledge?" The answer that we have another purpose in mind besides exploring the ntinent of Geometry. We believe that, in learning geometry this new way, you will make two distinct kinds of gain—u will add to your knowledge of geometry and *you will rease your power to think clearly about matters which have thing to do with geometry.* No one can deny the intensely actical value of the ability to do effective reasoning.

## *The First Few Steps*

Let us examine a piece of reasoning. We have chosen an

extremely simple example, but it will serve our purpo admirably.

*All persons born in Canada are Canadian citizens.*
*Marion was born in Canada.*
*Therefore, Marion is a Canadian citizen.*

You will agree that if the first two statements are accept then we are bound to accept the conclusion.

Let us take another example of reasoning:

*All rattlesnakes giggle when they read the comic strips.*
*Ratalia is a rattlesnake.*
*Therefore, Ratalia giggles when she reads the comic strips.*

Absurd, you say. Of course, it is absurd, but compa this argument with the first one. What have they in commo This—if we accept the first two statements, we cannot rej the conclusion. Both conclusions are inescapable once t preliminary statements are accepted.

You may object, saying, "Who ever heard of a rept reading 'Palsie Walsies'? A shrewd remark! But we did n say that the conclusion was *true*. Let us repeat what actually said: *if the first two statements are accepted, the co clusion must also be accepted.* It is possible to reason correc about matters of whose truth we know precisely nothi Here is another example in which the $X$'s, $Y$'s and $Z$'s a three groups of things:

All $X$'s are $Y$'s.
All $Y$'s are $Z$'s.
Therefore, all $X$'s are $Z$'s.

The first two statements in each of our three argumen are called ***assumptions.***

All of us make assumptions. When you set out for scho this morning, you assumed, or took for granted, that y would require the usual time—a second or two longer than t world's record for the half mile. As you know, this assum

n sometimes does not hold good! There were other assumptions which you did not recognize as such; for example, the umption that a holiday had not suddenly been declared. person who stumbled when leaving an elevator assumed at the elevator floor was at the same level as the floor at ich he was getting off. Except when asleep, we make ny assumptions in every hour. Some are built on a foundation close to the bed-rock of truth as man is able to go; some lance perilously on bases less substantial than a toy balloon.

Assumptions are often so deeply embedded in a discussion at, either they remain concealed, or are exposed only with ficulty. It has been said that the ability to detect hidden umptions is a sign of high intelligence. Find out what u can do with the following exercises.

### EXERCISES

int out one or more assumptions in each of the following:

1. "Of course he will be tall. Look at the length of his t."

2. "Is this soap good! Why, we sell more of it than of any ier brand."

3. "He lives four miles from Harriston and I live six miles m it. So we are only two miles apart."

4. There were ten problems on the paper. John had eight fect solutions. Hence, his percentage will be at least hty.

5. In seventy-five per cent of the automobile accidents t year in Sumburg, the drivers were men. Clearly, the mburgian women motorists are more expert than the men.

6. "You need have no hesitation in buying nationally vertised articles."

7. "If you remain in my employment, you may expect a sonable rise in salary next year."

**8.** "First we obtained the points 0 and 100 on the ste Then we divided the distance between these points into hundred equal parts, thus graduating the stem in degrees."

**9.** "I know that anhydrous copper sulphate acquires blue colour when brought into contact with water. When t anhydrous copper sulphate touched the raw potato, a bl colour was obtained. Hence, the potato contained water."

**10.** "Since $ax = ab$, $x = b$."

**11.** Two lines cannot intersect in more than one point.

**12.** Two lines cannot enclose a space.

**13.** Each of the angles is a right angle. Hence, they a equal.

## *Speaking the Same Language*

Let us glance at two more arguments:—
*All pitchers will break if they are dropped.*
*Tom is a pitcher.*
*Therefore, Tom will break if he is dropped.*

* * *

*All rulers are instruments for measuring length.*
*The king of Ruritania is a ruler.*
*Therefore, he is an instrument for measuring length.*

Obviously, there is a difficulty. Express it in your o words * * * You see that confusion can arise reasoning if we are not careful about our meanings. In first argument, the word 'pitcher' is used in one sense in first assumption, and in a quite different sense in the secon

Let us suppose that the following question is asked o group of pupils some of whom are Canadians and some Engli

*A ton of wheat costs* \$44.80. *What is the cost per pound cents?*

The Canadians say that the answer is 2.24 cents; the glish stoutly maintain that the answer is 2 cents. What s caused the disagreement? Simply this: the Canadian ton ighs 2000 pounds, and the English ton, 2240. The word n' has two distinct meanings. Before doing the calculation would have been wise to state whether the ton was English Canadian, the 'long' ton or the 'short' one. What was ded at the very outset was a *definition* of the term 'ton'.

*Crooked thinking is often caused by the use of words so familiar that we do not know how vague and sketchy to us their meaning really is. Just as countless disputes have flared up between families and between nations by reason of ill-defined boundaries, so confusion, controversy and bad feeling have arisen times without number because the territory enclosed by a word varies from one group to another, from person to person, or in the mind of the same person from one time to another.*

In matters of law, important decisions often rest on the cise meaning attached to very common words which, it is en assumed, convey the same meanings to different persons. r instance, in certain codes of law, a distinction between usebreaking' and 'burglary' is that the former cannot be mitted at night, nor the latter during the day. What is ght'? Is it the period between sunset and sunrise? If so, would be necessary to decide whether daylight saving or ndard time would govern the duration of night in certain ces during the summer. In English law, night "for all poses relating to larceny or other similar offences including glary" means the period from 9 p.m. to 6 a.m. Statutes ling with other matters such as poaching and the operation factories give different definitions. In aerial transportation Canada, night is the period between half an hour after sunset l half an hour before sunrise, except for 55° N latitude l further north, where a different definition is employed.

We shall see that, in our study of geometry, the utmo care is taken to prevent the confusion that arises out of vari tions in meaning. Let us remember to use similar care our discussions and in our thinking when school days are pa and geometry is a blurred, if pleasant, memory.

### EXERCISES

1. (*a*) Why are definitions important in games? (*b*) Gi examples of definitions in three different games.

2. State two meanings that could be taken from t sentence: "How horrid that bark is!"

3. Give the meaning that would be conveyed by t following sentence to (*a*) a Canadian, (*b*) an Englishma "Amyl is a homely girl."

4. Can the meaning of a word be affected by the tone voice in which it is uttered? Illustrate your answer reference to the following quotations: "That's a very pleas report from your principal"; "Brutus is an honourable ma "Bill likes onions".

## *If . . . . then . . .*

Let us make the following two assumptions:

*All cats make fiendish noises.*
*My pet, Marco Polo, is a cat.*

From them we obtain by correct (or logical) reasoning, conclusion: *Marco Polo makes fiendish noises.*

The whole argument could just as well have been cas the *'If . . . then . . .'* form as follows:

*If all cats make fiendish noises, and*
*if my pet, Marco Polo, is a cat,*
*then Marco Polo makes fiendish noises.*

An advantage of the 'If . . . then . . . ' form is that eminds us that the assumptions may, or may not, be true.

It may, or may not, be true that *all* cats make fiendish noises. ne cats, possibly including M. Polo himself, may have had ir vocal chords (or whatever part of their anatomy is ponsible) removed. When using the 'If . . . then . . . ' m, we are likely to remember that in a logical argument the th or falsity of the assumptions affects the truth or falsity the conclusion.

### EXERCISES

**1.** Express in the 'If . . . then . . . ' form the following tements. (The word 'then' may be omitted, but under- od.)

(*a*) Persons under the age of 16 years cannot obtain an omobile driver's licence.

(*b*) All white pines have five needles in each cluster.

(*c*) All straight angles contain 180°.

(*d*) Lines perpendicular to each other are at right angles.

(*e*) Provided that two angles are equal, their complements equal.

(*f*) Equal angles have equal supplements.

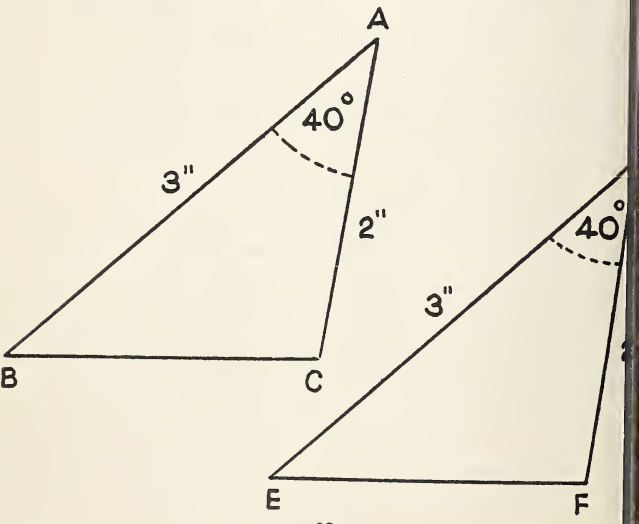
**2.** Write an argument in which, by logical reasoning, a e conclusion is drawn from false assumptions.

CHAPTER VI

# AN INTRODUCTION TO PROOF

*"I have found a very great number of exceedingly beautiful theorems."*
Fermat

A medical research worker claimed that a certain drug h important curative power. Before his claim could be accepte proof was needed. . . . Can this alloy meet certa specified requirements? Is this man really the hit-and-r driver the police believe him to be? Were these leade guilty of crimes against humanity? Is this signature a forger Was that fire accidental or the work of an incendiary? W that player eligible or ineligible? . . . On nearly eve



y of our lives, we hear or read of 'proof'. By studying ometric proof, we can learn some of the requirements which satisfactory proofs must fulfil.

Before leading into the matter of mathematical proof, we all ask you to do a simple geometric construction.

*Try to construct a copy of a triangle if you know two sides and angle between them.* Let the known sides be $AB$ and $AC$ d the known contained or included angle be $A$. We shall ve this exercise with you in the belief that you can construct riangle $DEF$ which is a copy of the triangle $ABC$.

Compare the length of $EF$ with that of $BC$. Compare the e of $\angle E$ with that of $\angle B$ and the size of $\angle F$ with that of $C$.

* * *

You went to work with your instruments and found that a by could be made. In other words, if two sides and the luded angle of a triangle are given, another triangle can drawn with two sides and the included angle respectively ual to those of the first triangle. Moreover, you found (*a*) at the remaining side of the second triangle is equal to the naining side of the first triangle, and (*b*) that the remaining o angles of the second triangle are equal respectively to the naining two angles of the first. In addition, we shall tell u that the areas of the two triangles are the same. We y express all of this by saying that, when two triangles have o sides and the included angle of the one respectively equal two sides and the included angle of the other, then the angles are ***congruent*** or identically equal.

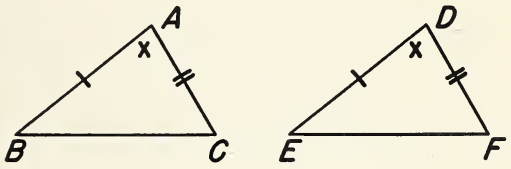
Your procedure was an experimental one. You measured remaining sides and the remaining angles in trying to find t if the triangles were congruent. This is an excellent thod. Similar methods are employed extensively in scien- c work. But, there are weaknesses. A certain set (or

certain sets) of measurements for the two sides and the co tained angle *might* be such that a copy of the triangle cou not be made. In your experimental work, you might n have encountered these contrary values. Again, can y be *certain* that the third side and the remaining angles of t one triangle are equal respectively to the corresponding si and angles of the other? Perhaps, if the sides had be measured to the thousandth of an inch, and the angles to t hundredth of a degree, it would have been found that, reality, they were unequal. Scientists have sometimes h to change their ideas when higher accuracy in measureme was attained.

Let us see if we can show, in some other way, that t triangles are congruent when two sides and the contain angle of one are equal respectively to two sides and the c tained angle of the other. Study this attempt very careful The explanations that follow it will answer most of y questions.

### PROPOSITION 1. THEOREM

***If two sides and the contained angle of one triangle e respectively equal to two sides and the contained gle of another triangle, then the triangles are con- uent.***




***Given:*** $ABC$ and $DEF$ are triangles in which $AB = DE$, $= DF$, and $\angle A = \angle D$.

***Required:*** To prove that triangles $ABC$ and $DEF$ are ngruent.

***Proof:*** Suppose that triangle $ABC$ is applied to triangle $EF$ so that the point $A$ falls on the point $D$, $AB$ lies along $E$, and $C$ falls on the same side of $DE$ as $F$.

Since $AB = DE$, (Given)
point $B$ must fall on the point $E$.

Since $\angle A = \angle D$, (Given)
line $AC$ must lie along $DF$.

Since $AC = DF$, (Given)
point $C$ must fall on the point $F$.

Since $B$ falls on $E$ and $C$ on $F$,
must coincide with $EF$.

Therefore, triangle $ABC$ coincides with triangle $DEF$.

Therefore, the triangles are congruent.

Note that, in proving the congruency of the triangles, we have proved that the remaining four magnitudes are respectively equal. 1. $BC = EF$; 2. $\angle B = \angle E$; 3. $\angle C = \angle F$; 4. Area of triangle $ABC$ = area of triangle $DEF$.

**Discussion of the foregoing demonstration.** Did you notice that, in the demonstration just given, *no reference was made to the actual lengths of the two sides, or to the number of degrees in the included angle?* The equality of $\angle B$ and $\angle E$, of $\angle C$ and $\angle F$, and of $BC$ and $EF$ were established *without measurement of these magnitudes.* The proof applies to *every* pair of triangles in which two sides and the included angle of the one are respectively equal to two sides and the included angle of the other.

> *We have not left ourselves open to the charge of rashness by jumping to a general conclusion after studying a number of examples.*
>
> *We have avoided the uncertainty always present in measurement.*

Now let us examine the proof in a more detailed manner.

The sentence in black type just above the figure consists of two parts:

1. It tells us what we are given. This is the 'if' part.
2. It tells us what we are to prove. This is the 'then' part.

The sentence is called the ***enunciation.***

Following the enunciation are the figures.

Examine the two paragraphs immediately after the figure. The first re-states what is given in the enunciation, but it does so by reference to the figures. The second re-states what the enunciation requires us to do, but it also refers to the figure. Together, these two paragraphs make up the ***particular enunciation.***

Then comes the actual proof. Notice that the authority r three statements is cited. Each of these assertions was gitimate because it was given in the enunciation.

* * *

Quite properly, it may be said that, in spite of the warning ven in our preliminary discussion of definitions, certain words d phrases were used without being defined. Let us collect em. They are ***point, line, triangle, angle,*** and ***contained gle.*** We shall consider them in turn.

**What is a point?** The famous geometer, Euclid, defined a int as *that which has no parts or magnitude.* Objection has en raised to this definition for two reasons:

1. A definition should preferably deal with properties ssessed by the thing defined, rather than with properties ich it lacks. For instance, *lack of bravery* would be a poor finition of *cowardice* unless 'bravery' is a more easily under- od term than 'cowardice'.

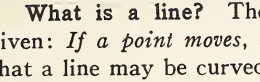
2. A definition should be reversible. For instance if a *d* is defined as a *feathered two-footed creature*, then a feathered o-footed creature should be a bird. The definition, *A air is an article of furniture on which to sit* is a poor one cause there are articles of furniture used as seats which are t chairs.

The definition of a point as that which lacks parts and agnitude is a statement of absent properties and it is also eversible. (The sum of zero dollars lacks parts and magni- de.)

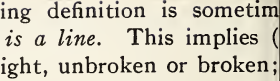
A more defensible definition of 'point' is ***that which has sition but no magnitude.*** In other words, it possesses a cation, but lacks length, breadth and width. Even if we ed the end of a bee's sting (compared with which a needle is crude, blunt affair), we could not make a point, because the ark would have some size. The pencil marks we call points e only attempts to represent them.

The second definition of a point is not above reproac the terms 'position' and 'magnitude' are no simpler than t term 'point'. Let us leave the term 'point' undefined. It unlikely to cause confusion.

**What is a line?** The following definition is sometim given: *If a point moves, its path is a line.* This implies ( that a line may be curved or straight, unbroken or broken; (See p. 99.)



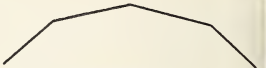
Representation of a curved line



Representation of a broken curved line



Representation of a straight line



Representation of a broken straight line

(2) that a line has length, but lacks breadth and thickness.

Another definition of a line is *that which has length, b neither breadth nor thickness.* This definition does not he us greatly because it prompts these questions: what is mea by length? by breadth? by thickness? Let us leave the term 'lin undefined and use these 'definitions' as working descriptior

**What is a straight line?** It has been said that *a straig line is one that lies evenly between its extremities.* But wh does 'evenly' mean? It would be as difficult to define it as define 'straight'.

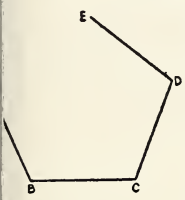
Another definition of a straight line is as follows: *A straig line is the shortest distance between two points.* This raises t question: What is meant by 'shortest distance'?

Other definitions have been proposed, but they also ha their weaknesses. Let us leave the phrase *straight line* u defined. When the word 'line' is used in this book, it will

derstood that the reference is to an unbroken straight line. other lines are referred to, this will be made clear in the t.

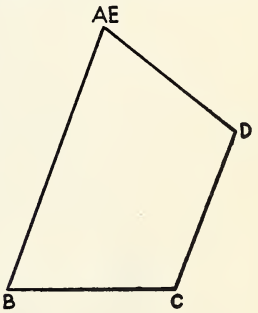
This epidemic of undefined terms may seem deplorable to u, but later you will see that some terms must be left un- ined.

**What is a triangle?** You will recall that if a number of connected straight lines are not in the same straight line, the figure is called a ***broken straight line.*** Thus, *ABCDE* is a broken straight line.




If *A* and *E* were to coincide, the figure would be a closed one. A closed broken line is called a ***polygon,*** which means literally, 'many angles'. (Gk. *polus*, ny; *gonia*, angle.) ***A triangle is a polygon of three sides.***

**What is an angle?** An angle has en described (not defined) as an ount of rotation. The following scription is sometimes given: *an gle is formed by two intersecting ight lines.* The point at which arms of an angle meet is called ***vertex*** (plural, *vertices*). How ny vertices are there in a triangle?




view of the difficulty of giving a satisfactory definition of gle', the term will be included among those left undefined.

**What is a 'contained' (or 'included') angle?** An angle is d to be 'contained' by, or 'included' between, its arms.

There are three other terms which should be considered.

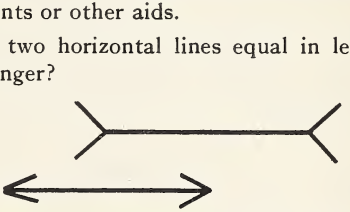
In a ***theorem,*** we are given certain information (or dat and are required to prove a certain statement or conclusion.

In a ***problem,*** we are given certain information a required to make a geometrical construction.

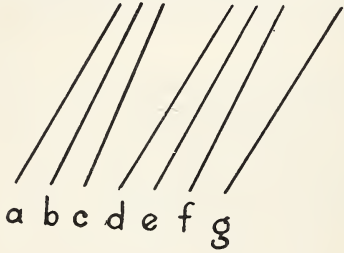
A ***proposition*** is a complete statement of (*a*) what given, (*b*) what is to be proved or constructed.

**A digression on the 'obvious'.** Do this little quiz witho using instruments or other aids.

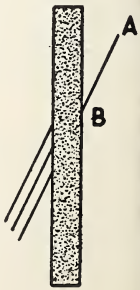
1. Are the two horizontal lines equal in length? If n which is the longer?



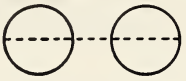
2. Which line, if any, is parallel to the line marked *a*?

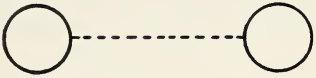



3. If *AB* were produced, along which of the three lines would it lie?

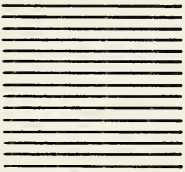


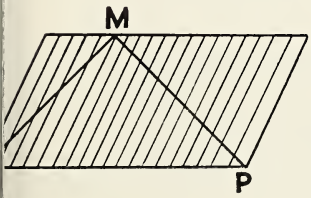
4. Which of the two dotted lines is the longer?





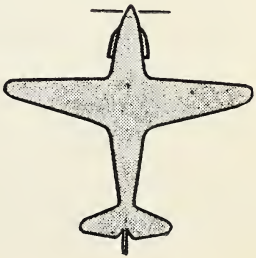
5. Does the outline form a square? If not, which is longer, length of the lines or the depth of the figure?






6. Is $MN$ equal to $MP$?

7. Is the wing-span greater than over-all length of the aeroplane?



It is told of the mathematician Laplace that he spe almost an hour trying to understand one of his own proo In this proof, he had lightly passed over some of the steps wi an "It is easy to see. . . "

Some modern mathematicians have devoted years to t scrutiny of statements which earlier workers accepted u questioningly. " 'Obvious' is the most dangerous word mathematics."

> *Perhaps we have now made the point that, in mathematics, one should not jump to conclusions. Let us add quickly that this caution should not be confined to mathematics. It should be displayed equally in everyday affairs.*

* * *

Now let us turn our attention to the assumptions in Propo tion 1. What was given in the enunciation is, of course, assumption. To be explicit: it was assumed that two triang were such that two sides and the included angle in one w respectively equal to two sides and the included angle in t other. Note that this assumption was made about the t triangles *in their original position.*

In our proof, we did not keep the triangle $ABC$ in original position; we brought it over to the triangle $DE$ How do we know that some or all of the seven magnitudes the triangle $ABC$ were not altered? This may seem ridiculo but remember our "Digression on the Obvious". The weig of a body is changed when the body is brought down int mine or carried up a mountain. The magnetic properties

iron bar are not the same when it is lying in the N-S direc-
n as when it is in the E-W direction. We shall *assume* that
ne of the seven magnitudes of the triangle $ABC$ was altered
the movement.

Also, we assumed that, if $B$ falls on $E$, and if $C$ falls on $F$,
n the straight lines $BC$ and $EF$ coincide completely. This
ms reasonable; it accords with our everyday experience.
t, if we leave the phrase 'straight line' undefined, we have
de the assumption mentioned.

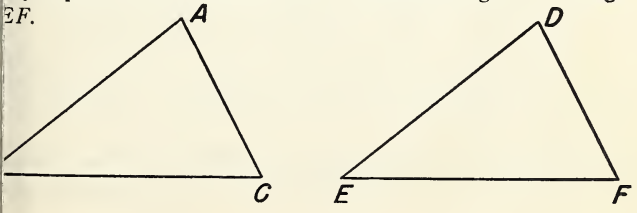
This does not exhaust the assumptions made. It is not
ended that you should memorize these particular assump-
ns for the sake of being able to reproduce them on an
mination paper. Our purpose is achieved if we have shown
t assumptions are as necessary in a proof as they are when
1 set out for school in the morning. It is hoped that you will
imagine that only mathematical arguments require minute
utiny.

> *The definitions involved, and the assumptions made, in all important discussions should receive the keen critical examination to which mathematical discussions are subjected.*

**EXERCISES**

(The protractor may be used in Exercises 2-8)

**1.** In each of the following cases, decide if you are given that
o sides and the contained angle in triangle $ABC$ are respec-
ely equal to two sides and a contained angle in triangle
$EF$.



(*a*) $BC = DE$; $CA = DF$; $\angle C = \angle D$.
(*b*) $AB = EF$; $AC = DF$; $\angle B = \angle E$.
(*c*) $\angle B = \angle E$; $AB = EF$; $BC = DE$.
(*d*) $\angle A = \angle F$; $\angle B = \angle E$; $AC = DF$.
(*e*) $AB = EF$; $BC = DE$; $CA = DF$.
(*f*) $AB = EF$; $\angle A = \angle F$; $AC = DF$.

**2.** At the point $A$ in a straight line $AB$ of length $2\frac{1}{2}$ i make an angle $BAX$ of 40°. From $AX$ cut off $AC$, 2 in. length. Join $BC$ and measure the line.

**3.** Construct a triangle $ABC$ from the following data: $AB = 6$ cm., $BC = 4$ cm., $\angle B = 47°$.

**4.** Construct a triangle $ABC$ in which $BC = 7.5$ cr $CA = 6.2$ cm., $\angle C = 58°$.

**5.** Construct a triangle $ABC$ in which $BC = 3$ in., $AB =$ in., $\angle C = 45°$. Measure and record the length of $AC$. C you obtain more than one triangle from the data?

**6.** Construct a triangle $ABC$ in which $BC = 3$ in., $AB$ 2.1 in., $\angle C = 45°$. Measure and record the length of $A$ Can you obtain more than one triangle from the data?

**7.** Try to construct a triangle $ABC$ in which $BC = 3$ i $AB = 1.7$ in., and $\angle C = 45°$. Wherein does your difficul lie?

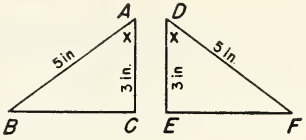
Note that in Exercises 5 and 6 you have been given t sides of a triangle and an angle not included by them.

**8.** Construct a triangle $ABC$ in which $BC = 2.1$ in., $CA$ 3.1 in., $\angle A = 30°$. How many triangles can be drawn fr the information given?

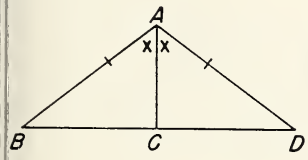
**9.** (*a*) In the triangles $ABC$ and $DEF$, are two sides a the included angle in one equal to two sides and the includ

gle in the other?

(*b*) Use the method of superposition (that is to say, the ethod of proof used in Proposition 1) to prove that the o triangles are congruent.




In Exercises 10 and 11, you need not go through the proof Proposition 1. If you can show that two triangles have o sides and the contained angle in one respectively equal to o sides and the contained angle in the other, you may say thout more ado, "Therefore, the two triangles are congruent."




**10.** (*a*) Certain data about the triangles *ABC* and *ACD* are indicated in the figure. Are the triangles congruent? Give your reasons.

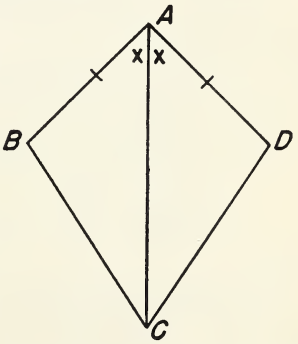
(*b*) What magnitudes, other than those indicated in the figure, are equal?

**11.** (*a*) Is the triangle *BC* congruent with the triangle *ADC*? Give your reasons.

(*b*) What magnitudes, her than those indicated in e figure, are equal?

(*c*) What relation does the le *AC* bear to the angle *AD*?

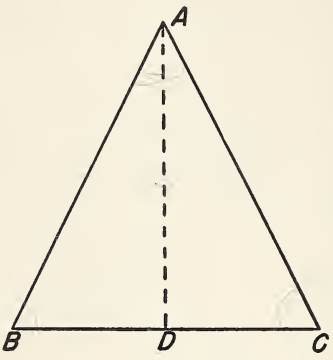
(*d*) What relation does *C* bear to the angle *BCD*?




DEFINITION: ***An isosceles triangle is one with at least o equal sides.***

### PROPOSITION 2. THEOREM

***If a triangle is isosceles, then the angles opposite th equal sides are equal.***




***Given:*** $ABC$ is a triangle in which $AB = AC$.

***Required:*** To prove that $\angle B = \angle C$.

***Proof:*** Let $AD$ be the line which bisects $\angle BAC$, and l $AD$ meet $BC$ at $D$.

In triangles $ABD$ and $ACD$,

$\begin{cases} AB = AC, & \text{(Given)} \\ AD \text{ is common to both,} & \\ \angle BAD = \angle CAD. & \text{(By construction)} \end{cases}$

Therefore, triangle $ABD$ is congruent with triang $ACD$, and $\angle B = \angle C$. (Two sides and a co tained angle, etc.)

**Discussion of Proposition 2.** It will be noted that, in th proof, we have made use of the conclusion of Proposition 1. Thu Proposition 1 is one of the foundation stones in Proposition Later, we shall refer to an important assumption made the proof.

## EXERCISES

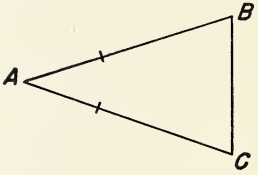
1. In the figure of Proposition 2, state specifically what agnitudes (other than the angles $B$ and $C$) have been proved ual.

DEFINITION: ***The line joining any vertex of a triangle the middle point of the opposite side is called a edian of the triangle.***

In doing the remaining exercises, you may make use of the nclusions of Propositions 1 and 2.

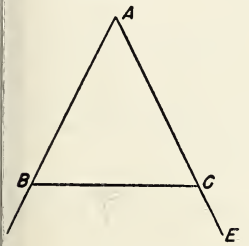
2. In the figure of Proposition 2, name a median of the iangle $ABC$. How many medians has a triangle?

3. Name the equal angles in e triangle $BAC$.



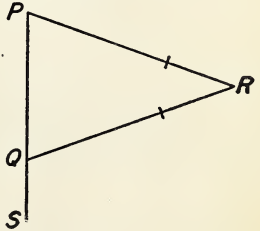

4. In the second figure, $AD$ $AE$ and $BD = CE$.

(*a*) Prove that the triangle $BC$ is isosceles.




(*b*) $AB$ is $5x-9$ units in length; $AC$ is $2x+3$ units; $BD$ is $\frac{x}{2}$ units. Find $x$ and, hence, the length of $CE$.

5. In the triangle $PQR$, $RP = RQ$; $\angle P = (5x-40)°$; $\angle PQR = (3x+4)°$; $PQ$ is produced to $S$. Find $x$ and, hence, the number of degrees in the angle $RQS$.




6. Show that, if a triangle is osceles, the supplements of the ıgles opposite the equal sides e equal. (You will recall that, the sum of two angles is 180°, ıch is the supplement of the ther.)

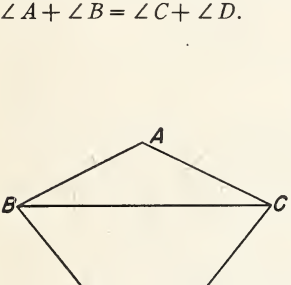
DEFINITION: ***An equilateral triangle is a triangle wit three equal sides.***

**7.** Show that, if a triangle is equilateral, its three angl are equal.

**8.** $ABC$ is a triangle in which $AB = AC$. $D$ is a poi within the triangle such that $DB = DC$. Prove that $\angle AB = \angle ACD$.

DEFINITION: ***A quadrilateral is a four-sided polygo***

**9.** $ABCD$ is a quadrilateral in which $AB = BC = CD = D$. Prove (*a*) that $\angle A = \angle C$, (*b*) that $\angle B = \angle D$, (*c*) th $\angle A + \angle B = \angle C + \angle D$.




**10.** $ABC$ and $DBC$ are tw triangles in which $AB = AC$ a $DB = DC$.

(*a*) Name three pairs of equ angles in the figure.

(*b*) Join $AD$. Prove that t triangle $ABD$ is congruent wi the triangle $ACD$.

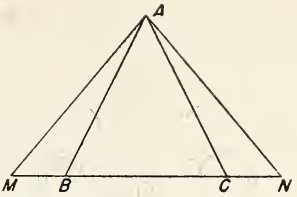
(*c*) Prove that $AD$ bise the angle $BAC$. What oth angle does $AD$ bisect?

(*d*) What object does t figure $ABDC$ resemble?

DEFINITION: ***A line joining two non-adjacent vertic of any polygon of four or more than four sides is said be a diagonal of the polygon.***

In the preceding figure, $BC$ is a diagonal of $BDCA$.

11. In the accompanying ure $AB = AC$, and $BM = CN$.

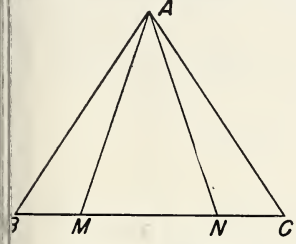



(*a*) Prove that the triangle $MN$ is isosceles. (*Hint*: Make e of Exercise 6.)

(*b*) Find four pairs of equal ies in the figure.

(*c*) Find five pairs of equal angles.

(*d*) Find two pairs of congruent triangles.




12. $ABC$ is a triangle in which $AB = AC$. The points $M$ and $N$ on $BC$ are such that $BM = CN$.

(*a*) What kind of triangle is $AMN$? Why?

(*b*) Find four pairs of equal lines in the figure.

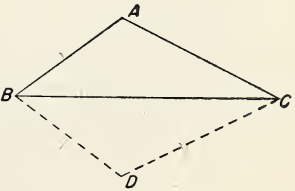
(*c*) Find five pairs of equal angles.

(*d*) Find two pairs of congruent triangles.

(*e*) From $A$ draw a median of the triangle $ABC$. Prove at it is also a median of the triangle $AMN$.

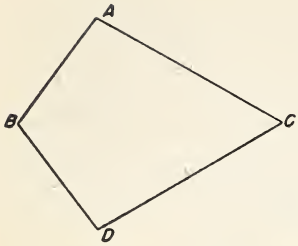
(*f*) Let the median drawn from $A$ meet $BC$ at the point $D$. $BN = 3x - 2$ units, $CM = x + 2$ units, and $MN = x$ units, find and, hence, the lengths of (i) $MD$, (ii) $ND$, (iii) $BD$, (iv) $CD$, ) $BM$, (vi) $CN$.

13. The triangle $ABC$ was rned about $BC$ as hinge til it took up the position $BC$. Draw a line which will vide the quadrilateral $BACD$ to two isosceles triangles.




(*a*) Prove that the triangles e isosceles.

(*b*) Name two diagonals of e figure $ABDC$.




14. In the quadrilater
$BACD$, $BA = BD$, and $CA$
$CD$.

(*a*) Draw a line which w
divide the quadrilateral in
two congruent triangles. Pro
that they are congruent.

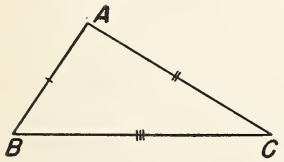
(*b*) Find three pairs of equ
angles. (*Hint:* Draw a diagonal

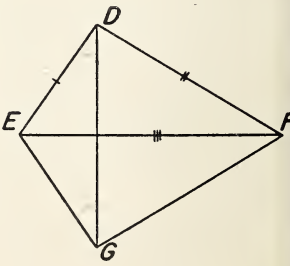
**Symbols and abbreviations.** We shall bring together som
of the symbols with which you are already familiar and sha
add some new ones.

= equals, is equal to, are equal to.
≡ is congruent with.
∠, ∠s angle, angles.
△, △s triangle, triangles.
∴ therefore.
⊥, perpendicular, is perpendicular to.

### PROPOSITION 3. THEOREM

***If the three sides of one triangle are respectivel
equal to the three sides of another triangle, the triangl
are congruent.***



***Given:*** $ABC$ and $DEF$ are two triangles in which $AB = E$, $AC = DF$, and $BC = EF$.

***Required:*** To prove that $\triangle ABC \equiv \triangle DEF$.

***Proof:*** Apply $\triangle ABC$ to $\triangle DEF$ so that $B$ falls on $E$, nd so that $BC$ lies along $EF$.

Since $BC = EF$, (Given)
must fall on $F$.

Let $A$ take up the position $G$ on the side of $F$ remote from $D$.

Join $DG$.

In $\triangle EDG$, $ED = EG$. (Given)
$\therefore \angle EGD = \angle EDG$. (In an isosceles $\triangle$, the angles opposite the equal sides are equal.)

In $\triangle FDG$, $FD = FG$. (Given)
$\therefore \angle FGD = \angle FDG$. (In an isosceles $\triangle$, etc.)
$\angle EGD + \angle FGD = \angle EDG + \angle FDG$.
$\therefore \angle EGF = \angle EDF$.
That is, $\angle BAC = \angle EDF$.

In $\triangle$s $ABC$, $DEF$,
$AB = DE$, (Given)
$AC = DF$, (Given)
$\angle BAC = \angle EDF$, (Proved)
$\therefore \triangle ABC \equiv \triangle DEF$. (Two sides and contained angle)

**Discussion of Proposition 3.** You have probably noted at, in the proof, use was made of the conclusions of Proposi- ns 1 and 2. We have forged a little chain with three links.

Note carefully the following lines in the proof:—

$$\angle EGD = \angle EDG.$$
$$\angle FGD = \angle FDG.$$
$$\therefore \angle EGD + \angle FGD = \angle EDG + \angle FDG.$$

You will recall that, when solving an algebraic equati such as $2x+8 = x+12$, it is permissible to add the same numb to each side. If we write the process in full, we have (addi 5 to each side)

$$2x + 8 = x + 12,$$
$$5 = 5,$$
$$\therefore 2x + 8 + 5 = x + 12 + 5.$$

In the three lines of the proof to which attention has be directed, we have made use in geometry of a procedure famili to you in algebra. The right to perform this operation *assumed.* In the following statement of the assumption, equ magnitudes are referred to as 'equals'.

***Assumption 1***

***If equals are added to equals, the sums are equal.***

You are reminded of another procedure used in dealing wi equations.

$$2x + 8 = x + 12.$$
$$8 = 8.$$
$$\therefore 2x + 8 - 8 = x + 12 - 8.$$

Here we subtracted equals from equals and assumed th the remainders are equal. This assumption, like the p ceding one, is often used in geometry.

***Assumption 2***

***If equals are subtracted from equals, the remainders are equal.***

For convenience in referring to these assumptions, we ve given them numbers.

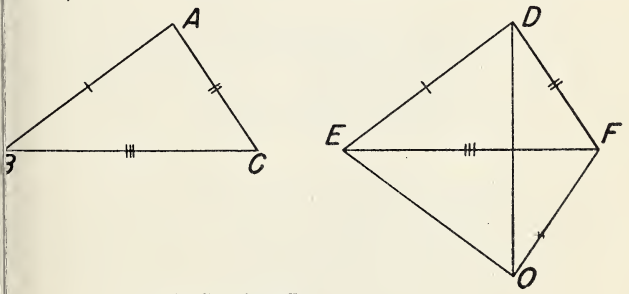
### EXERCISES

State which of the two assumptions discussed in the im- diately preceding paragraphs is used in each of the following:

1. If $a = b$ and $c = d$, then $a + c = b + d$.
2. If $m = n$ and $p = q$, then $m - q = n - p$.
3. If $m = 2$ and $m = n$, then $n - 2 = 0$.
4. If $p + q + r = 8$, $q = 3$, and $r = 4$, then $p = 1$.
5. If $c = d$, then $2c = 2d$.
6. If $\angle A = 30°$ and $\angle B = 150°$, then $\angle A + \angle B = 180°$.
7. If $\angle C + \angle D = 90°$ and $\angle D = 30°$, then $\angle C = 60°$.

## *Brighter Geometry*

eing an extract from Walrus and Carpenter's "Very New ometry for Schools", with profound apologies to Lewis rroll[1].)




Now ABC and DEF
Were lying close at hand;
Their sides were equal, each to each,
They were so neatly planned,
"If we were only congruent,"
They said, "It would be grand."

[1] From the *Boys' Own Paper*—by kind permission of the publishers, terworth Periodicals, Limited, Redhill, Surrey, England.

"If they should put us base to base,
And let A fall at O,
Do you suppose," said DEF
"That they could prove us so?"
"I doubt it," answered ABC,
"With brains so dull and slow."

"But don't you think that they would see,"
The former did beseech,
"If B exactly fits on E,
That C to F will reach?
It must be so when all our sides
Are equal each to each."

"And then it only needs a line
Connecting O with D,
To make the argument become
As plain as ABC."
("That hardly is," the latter said,
"A compliment to me.")

But DEF went on, "They ought,"
He said, "To grasp with ease,
That DEO and DOF
Are both isosceles,
And hence the angles at their base
Are similar as peas."

"And here comes in a maxim which
The late E. Uclid said,
That seems to hit," said DEF,
"The nail upon the head."
("I grieve to hear," wept ABC,
"The gentleman is dead.")

"He died, alas, but not before
He made his great decree,
'If equals are to equals put
The wholes shall equal be';
Which demonstrates beyond a doubt
That O must equal D."

"So, FEO and FED,
'Tis now as plain as . . . plain,"
("You nearly," murmured ABC,
"Were rude to me again.")
"Have got those angles equal which
Their equal sides contain."

"The two are therefore congruent
And FED is me,
While FEO is really you . . ."
("I much," said ABC,
"Object to that objective case
After the verb 'to be'.")

But DEF cared nought for verbs,
To be or not to be,
"I'm congruent with you," he cried,
"And so are you with me;
Thus there is nothing left for us
To say but Q. E. D."[2]

**A tricky little point.** Consider the statement, "All acute gles are less than one right angle". Let us turn it around l say, "All angles less than one right angle are acute angles."

Each of these two statements is called the *converse* of the ıer.

Instead of saying, "All acute angles are less than one right gle", we may use the 'If . . . then' form: "If an angle is ıte, then it is less then one right angle." Express the ıverse in the 'If . . . then' form.

In our example, both the original statement and its con-se are true. Now consider these assertions: "All right gles are equal", "All swimmers are ducks", and "All turkeys red-heads". Are the converses of these statements true?

[2] Q. E. D., an abbreviation for *Quod erat demonstrandum* (Which was to oroved), was formerly written at the end of the proof of a theorem.

*If a statement is true, its converse may be true or it may be false.*

*You should be as careful about converses in everyday life as you are about converses in mathematics.*

**EXERCISES**

**1.** To the best of your knowledge, describe each statem and also its converse as *true* or *false*.

(*a*) All numbers that end in 0 are exactly divisible by 5

(*b*) If you are standing on your head, your face beco red.

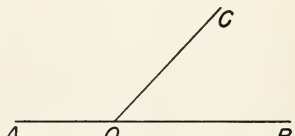
(*c*) All daughters of sisters are cousins.

(*d*) All great-grandfathers are men.

(*e*) All bipeds are human beings.

(*f*) All negroes possess wavy hair.

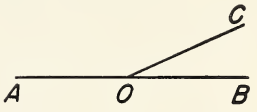
(*g*) If a man is of good character, he does not steal.




(*h*) If $AB$ is a straight line, then $\angle BOC + \angle COA = 1$

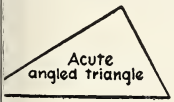
**Another assumption.** It will be recalled that we have cribed a straight angle as half a complete revolution. Thus, $OA$ and $OB$ are in a straight line, $AOB$ is a straight angle.

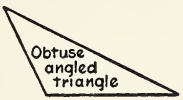
e converse of this statement hat if $\angle AOB$ is a straight gle, then $OA$ and $OB$ are in a aight line. As you are aware *onverse must be proved inde- dently of the original state- nt.* For instance, from the tement, "If the road is icy, it is slippery", we cannot infer converse, "If the road is slippery, it is icy".




Now the converse, "If $\angle AOB$ is a straight angle, then $OA$ l $OB$ are in a straight line" *can* be proved independently of statement, "If $OA$ and $OB$ are in a straight line, $\angle AOB$ is traight angle". We shall not give the proof, but shall ask u to accept the converse as an assumption. (See p. 82.)

DEFINITIONS: ***An acute-angled triangle is a triangle in which the three angles are acute.***



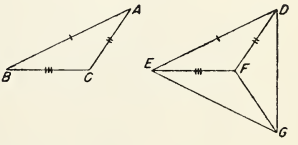




***An obtuse-angled triangle is a triangle which contains an ob- se angle.***

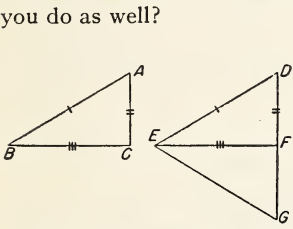
***A right triangle is a triangle which contains an angle 90°.***

## EXERCISES

1. $ABC$ and $DEF$ are two angles in which $AB = DE$, $= DF$, and $BC = EF$. The gles $ACB$ and $DFE$ are ob- se, that is to say, greater than ight angle. A pupil tried to



prove Proposition 3 using these obtuse-angled triangles. Wh he joined $DG$, he found that the line lay outside the triangl instead of inside as it was in the figure of Proposition 3. proved that the triangles $ABC$ and $DEF$ are congruent, b he used Assumption 2 (page 112) instead of Assumption 1. C you do as well?




**2.** $ABC$ and $DEF$ are t triangles in which $AB = D$ $AC = DF$, and $BC = EF$. T angles $ACB$ and $DFE$ right angles. Prove that t triangles are congruent.

*Hints*: **1.** What is the sum of $\angle$s $DFE$ and $GFE$?

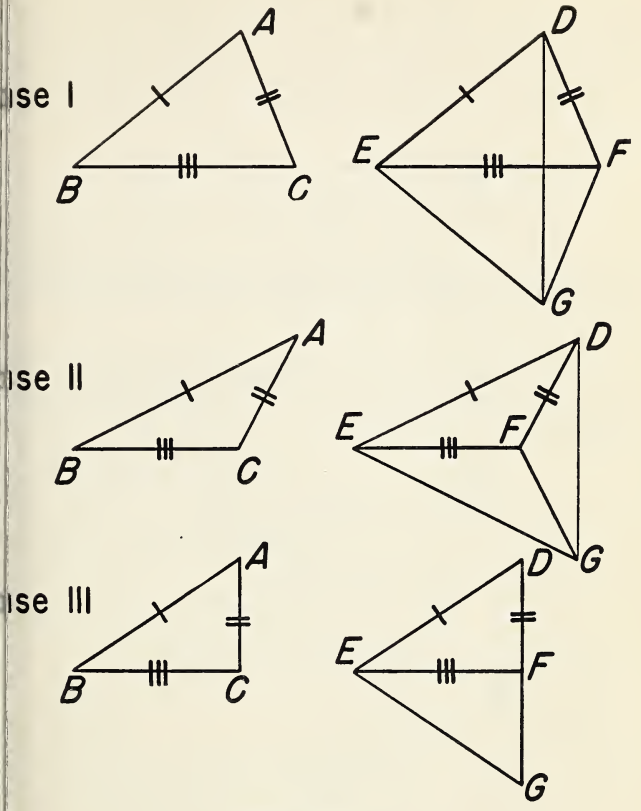
**2.** What kind of angle is $\angle DFG$?

**3.** What can be asserted of the two lines $DF$ and $F$

**4.** Is the figure $EDFG$ a quadrilateral or a triang Give your reasons.

* * *

Let us return to Proposition 3. When drawing the figu we used two acute-angled triangles. But we were not requir (in the enunciation) to confine ourselves to a special kind triangle. We were asked to prove the congruency of *any* p of triangles with the three sides of the one respectively eq to the three sides of the other. The two preceding exerci show that obtuse-angled and right triangles should also ha been taken into account. In brief, in a complete proof Proposition 3, the following three cases should be considered:




**EXERCISES**

1. Construct a triangle $ABC$ in which $AB = AC = 7$ cm, $\angle A = 60°$. Measure and record the size of (*a*) $BC$, (*b*) $\angle B$, (*c*) $\angle C$.

2. Construct a triangle $ABC$ in which $AB = 6.5$ cm.,

$BC = 5.3$ cm., and $CA = 4.7$ cm. Measure and record t size of $\angle C$.

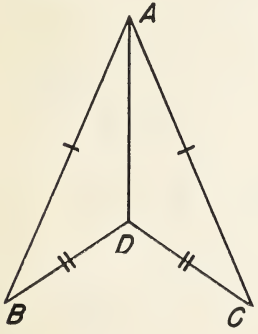
3. $ABC$ is a triangle in which $AB = AC$. $D$ is the mid point of $BC$.

(*a*) Prove that $AD$ bisects the angle $BAC$.

(*b*) Prove that $AD$ is perpendicular to $BC$.

4. $ABC$ is a triangle in which $AB = AC$. The bisector of the angle $A$ meets $BC$ at $D$. Show (*a*) that $AD$ is perpendicu to $BC$, (*b*) that $BD = DC$.

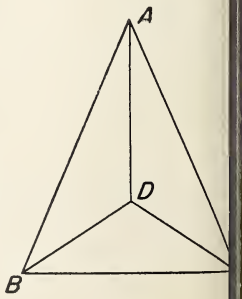
5. The opposite sides of a quadrilateral are equal. Pr that each diagonal divides the quadrilateral into two congru triangles.




6. In the first figure, $AB = A$ and $DB = DC$.

(*a*) Prove that $\angle ABD$ $\angle ACD$.

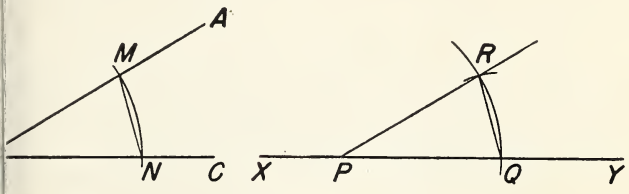
(*b*) If $AD$ is produced to prove that $\angle BDE = \angle CDE$.




7. In the second figure, $AB = AC$, $AD$ bisects $\angle BAC$; $BD$ bisects $\angle ABC$. Prove that $CD$ bisects $\angle ACB$.

### PROPOSITION 4. PROBLEM

***At a given point in a given straight line, construct angle equal to a given angle.***




***Given:*** $P$ is the given point in $XY$ the given straight line. $C$ is the given angle.

***Required:*** At $P$ draw a line which with $XY$ will include given angle.

***Construction:*** With centre $B$ and any radius, draw an arc ing $BA$ at $M$ and $BC$ at $N$.

With centre $P$ and the same radius, draw an arc cutting $'$ at $Q$.

With centre $Q$ and radius $NM$, draw an arc cutting the arc n centre $P$ at $R$.

Join $PR$.
Then $RPQ$ is the required angle.

***Proof:*** Join $MN$, $RQ$.
In $\triangle$s $RPQ$, $MBN$,

$$\left\{\begin{array}{ll} RP = MB, & \text{(Construction)} \\ PQ = BN, & \text{(Construction)} \\ QR = NM. & \text{(Construction)} \end{array}\right.$$

$\therefore \triangle RPQ \equiv \triangle MBN$, (Why?)
and $\angle RPQ = \angle MBN$.

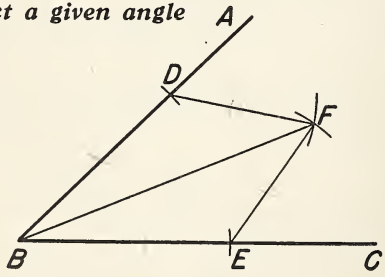
**Discussion of Proposition 4.** This problem, the f encountered among the four propositions dealt with so f could have been solved by you without assistance. ( purpose in giving the construction and proof in full is to sup you with a model for writing out the constructions and pr for other problems.

Notice that, following the particular enunciation, came method of construction. This was *not* followed by measu ment of the angles to find if the procedure followed was corr Had this been done we would have had some doubts. Inst of measuring the angles, we proved that the angle construc was equal to the angle given. In so doing, we made use c *previously* proved proposition, the third.

Observe also that each step in the proof was defended justified. For instance, the statement "$RP = MB$" justified by the word 'Construction'; the two lines were eq because we made them equal.

### PROPOSITION 5. PROBLEM

***To bisect a given angle***




***Given:*** $ABC$ is the given angle.

***Required:*** Bisect $\angle ABC$.

***Construction:*** With centre $B$ draw an arc cutting $BA$ $D$ and $BC$ at $E$.

With centre $E$, draw an arc.

With centre $D$ and the same radius, draw another arc cutting first at $F$.

Join $BF$.

Then $BF$ is the bisector of $\angle ABC$.

***Proof:*** Join $FD$, $FE$.

In $\triangle$s $BDF$, $BEF$,

$BD = BE$, (Construction)

$DF = EF$, (Construction)

$BF$ is common.

$\therefore \triangle BDF \equiv \triangle BEF$, (Why?)

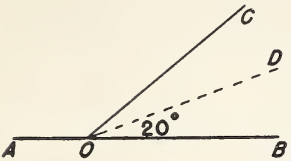
and $\angle DBF = \angle EBF$.

* * *

It may now be pointed out that, in Proposition 2, it was *umed* that the bisector of an angle could be found.

**The trisection of an angle.** For centuries, mathematicians d to construct with compasses and straight edge a square al to a given circle. Another of the Three Famous Problems antiquity was to trisect any angle. We say *any* angle ause it is possible to trisect with compasses and straight e a number of special angles. It is known now that this blem of general trisection cannot be solved with the two ruments mentioned. However, if the restriction is re- ved, any angle can be trisected with little difficulty. It y be mentioned that some non-mathematicians are still ng to perform the construction with straight edge and ipasses only.

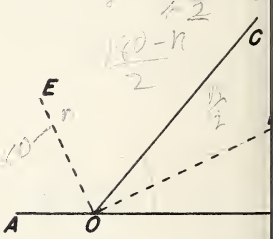
## EXERCISES




1. In the figure, $AB$ is straight line and $OD$ is the bis tor of the angle $BOC$. Calcul the size of the angle $COA$.

2. If in the preceding exercise, the angle $BOD$ was calculate the size of the angle $COA$.

3. In the figure, $AB$ is a straight line, and $OD$ and $OE$ are respectively the bisectors of the angles $BOC$ and $COA$.




($a$) If the angle $BOC$ is 40°, calculate the size of the angle $DOE$.

($b$) If the angle $BOC$ is $n$°, calculate the size of the angle $DOE$.

($c$) Give a formal proof of the statement that, irrespect of the size of the angle $BOC$, the angle between the bisect $OD$ and $OE$ is a right angle.

4. Draw any two unequal angles $ABC$ and $DEF$. C struct ($a$) an angle equal to their sum, ($b$) an angle equal their difference, ($c$) the bisector of their sum constructed ($a$), ($d$) the bisector of their difference constructed in ($b$).

5. With your protractor, make an angle of 178°. Us compasses and straight edge, bisect the angle. Measure two parts.

6. Bisect a straight angle.

If you can perform this construction, and can prove that really have bisected the angle of 180°, you have dealt cessfully with the next proposition.

### PROPOSITION 6. PROBLEM

***To draw a perpendicular to a given straight line at a en point in the line.***

### EXERCISES

(Use compasses and straight edge only)

1. Draw a horizontal line $AB$ about 4 inches long. Select points, $P$ and $Q$, on $AB$ about 2 inches apart. At $P$ struct a line perpendicular to $AB$ and extending above $AB$. $Q$ construct a line perpendicular to $AB$ and extending w $AB$.

2. Draw a vertical line $AB$ and select two points on it. ne of these points draw a horizontal line extending to the t of $AB$, and at the other point construct a horizontal line nding to the left of $AB$.

3. Construct an angle of (*a*) 45°, (*b*) $22\frac{1}{2}$°, (*c*) $67\frac{1}{2}$°, (*d*) $112\frac{1}{2}$°, 135°, (*f*) $157\frac{1}{2}$°, (*g*) 315°.

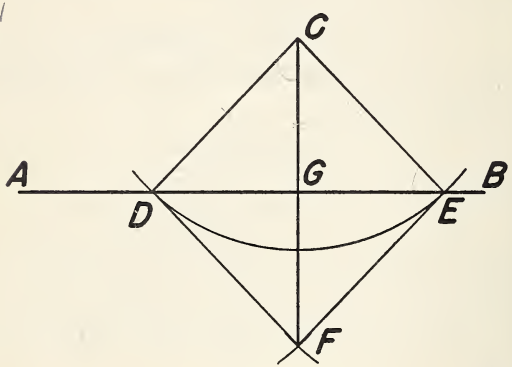
4. At one end of a straight line, draw a perpendicular to line.

5. Construct a triangle $ABC$ in which the angle $B$ is 90°, is equal to a given line, and $BC$ is equal to another given

6. Draw a line $AB$ 3 inches long. At $A$ and B, respectively, truct perpendiculars $AX$ and $BY$, each 3 inches long. Join Suggest a name for the figure $ABXY$.

### PROPOSITION 7. PROBLEM

***To draw a perpendicular to a given straight line fr a point outside the line.***




***Given:*** $C$ is the given point outside the given straight $AB$.

***Required:*** To draw from $C$ a perpendicular to $AB$.

***Construction:*** With centre $C$, draw an arc intersec $AB$ at $D$ and $E$. With $D$ and $E$ as centres, draw arcs of e radii intersecting at $F$. Join $CF$ cutting $AB$ at $G$.

Then $CG$ is the required perpendicular.

***Proof:*** Join $CD$, $CE$, $FD$ and $FE$.

In $\triangle$s $CDF$, $CEF$,

$CD = CE$, (Radii of same circ

$DF = EF$, (Radii of equal cir

$CF$ is common.

$\therefore \triangle CDF \equiv \triangle CEF$,

and $\angle DCG = \angle ECG$. (Why?

In $\triangle$s $CDG$, $CEG$,

$$\left\{\begin{array}{l} CD = CE, \\ CG \text{ is common}, \\ \angle DCG = \angle ECG. \end{array}\right. \quad \text{(Why?)}$$

$\therefore \triangle CDG \equiv \triangle CEG$ (Why?)

$\therefore \angle CGD = \angle CGE$,

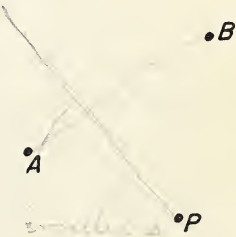
and $CG \perp AB$.

## EXERCISES

1. Draw any acute-angled triangle. From any vertex struct a perpendicular to the opposite side.

2. Draw any obtuse-angled triangle $ABC$ in which $C$ is the ise angle. From $A$, draw a perpendicular to $BC$.

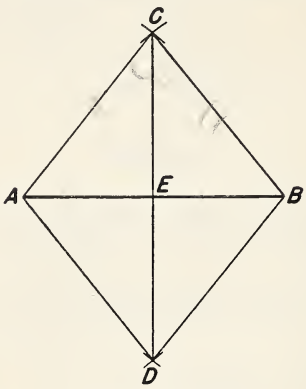
3. Draw any triangle. From a t outside the triangle draw perpen- lars to the sides.




4. $A$, $B$ and $P$ are three shrubs in a len. It is proposed to plant a th shrub $Q$ so that $PQ$ is bisected at t angles by $AB$. How can the re position of $Q$ be determined?

### PROPOSITION 8. PROBLEM

***To draw the right bisector of a given straight line***




***Given:*** $AB$ is the given straight line.

***Required:*** To construct the right bisector of $AB$.

***Construction:*** With $A$ and $B$ as centres and equal r draw arcs intersecting at $C$ and $D$. Join $CD$.

Then $CD$ is the required right bisector of $AB$.

***Proof:*** Let $CD$ cut $AB$ at $E$. Join $CA$, $CB$, $DA$ and

In $\triangle$s $CAD$, $CBD$,

$CA = CB$, (Why?)
$DA = DB$, (Why?)
$CD$ is common.

$\therefore \triangle CAD \equiv \triangle CBD$, (Why?)
and $\angle ACD = \angle BCD$.

In $\triangle$s $CAE$, $CBE$,

$CA = CB$,
$CE$ is common,
$\angle ACE = BCE$. (Why?)

$\therefore \triangle CAE \equiv \triangle CBE$, (Why?)
and $AE = EB$.
Also $\angle AEC = \angle BEC$,
and $CE \perp AB$. (Why?)

Since $E$ is the middle point of $AB$, and $CD \perp AB$, $CD$ is e right bisector of $AB$.

**Discussion of Proposition 8.** What propositions were used the proof of Proposition 8? Were they proved previously?

Give a proof for the following construction substituted for e one in Proposition 8:—

With centres $A$ and $B$ and equal radii, draw arcs inter- :ting at $C$. With centres $A$ and $B$ and equal radii, but ferent from those used previously, draw arcs intersecting at Join $CD$.

What advantage has the construction used in the text over e substitute?

Perform the substitute construction again, but this time ange to have $C$ and $D$ on the same side of $AB$. Prove that ) produced is the right bisector of $AB$.

A pupil followed the construction of Proposition 8, but ınd that his arcs did not intersect. What was the difficulty? hat precaution must be observed to ensure that the arcs ll intersect?

**Trisection of a straight line.** In view of what had been d concerning the trisection of an angle, you may wonder it is possible to trisect a straight line with straight edge and mpasses. It is possible to divide a line, not merely into o or three equal parts, but into any integral number of ual parts. We shall deal with this matter later, but it uld not be contrary to the criminal code if you were to end some of your spare time in trying to trisect a straight e.

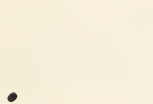
### EXERCISES

1. Using compasses and straight edge, divide a straig line into four equal parts.

2. How would you obtain a line the length of which is 75% the length of a given line?

3. Show that any point on the right bisector of the li joining two points is equidistant from the points.

4. $A$, $B$ and $C$ are the positions of three trees. It w required to find a point equidista from $A$, $B$ and $C$. How was t point obtained? If $A$, $B$ and were in the same straight li could a point equidistant from th be found? Explain.




5. Draw a circle which will pass through three given poi not in the same straight line.

DEFINITION: ***The circle which passes through t vertices of a triangle is called the circumscribed cir of the triangle.***

DEFINITION: ***If all the vertices of a polygon lie on a circ the polygon is said to be inscribed in the circle.***

In the figure of Exercise 5, the triangle formed by joining three points is inscribed in the circle.

6. Draw any triangle and its circumscribed circle. (D the triangle first!)

### PROPOSITION 9. PROBLEM

***To construct a triangle with sides of given length***

We shall leave this proposition with you as an exer Is it possible to perform the construction if you are given three lines?

## EXERCISES

**1.** Construct a triangle $ABC$ in which $AB = BC = 2$ in., d $AC = 2\frac{1}{2}$ in.

**2.** Construct a triangle $ABC$ in which $AB = BC = CA =$ n. What kind of triangle is $ABC$?

**3.** $ABC$ is a triangle in which $AB = AC$. If $AB =$ $-2)$ cm., $AC = (2x + 2)$ cm. and $BC = \left(\frac{x}{2} + 2\right)$ cm., nstruct the triangle.

**4.** Construct a quadrilateral $ABCD$ in which $AB = 2.5$ $BC = 2.3$ in., $CD = 1.9$ in., $DA = 2.1$ in., $\angle B = 41°$. easure and record the size of (*a*) the diagonal $AC$, (*b*) $\angle D$. y to do this construction without using the hints which ow.

*Hints*: Make a rough freehand drawing of $ABCD$, and te on it the given dimensions. Join $AC$. What are you en in $\triangle ABC$? Construct $\triangle ABC$ accurately. Measure . Write the length of $AC$ on the rough figure. What data ncerning $\triangle ACD$ are now available? Can you construct $ACD$ from these data? Construct $\triangle ACD$ accurately on ir carefully drawn $\triangle ABC$. Now you have learned how to nstruct a quadrilateral when given the four sides and one gle. You have seen that this involves two previously rned constructions of triangles.

**5.** Construct a quadrilateral $ABCD$ in which $AB = 4.8$ ., $BC = 5.3$ cm., $CD = 4.1$ cm., $DA = 5.0$ cm., $\angle A = 68°$. easure and record the size of (*a*) the diagonal $DB$, (*b*) $\angle C$.

**6. A class exercise.**

Each member of the class should construct an equilateral ngle. These should be of different sizes. Bisect the three gles. Examine the figures and see if these bisectors invariably s through one point. (When lines pass through one point,

they are said to be ***concurrent.***) If, in any triangle, t bisectors are not concurrent, the construction should performed a second time, more carefully. What probab inference may be made? (We say 'probable' because, in th exercise, you are depending upon observation, not proof.)

* * *

It is likely that, after inspection of all the figures, the follo ing conclusion has been drawn: it is probable that the bisect of the angles of equilateral triangles are concurrent.

Now draw triangles which are not isosceles. Inclu acute-angled triangles, right triangles, and obtuse-angl triangles. Draw the bisectors of the angles. Do they appe to be concurrent? What conclusion seems to be warrant now?

You see that we may enlarge the original conclusio Now we say that it is probable that the bisectors of the ang of *any* triangle are concurrent.

When a detective tries to 'reconstruct' a crime, he mu make a number of intelligent guesses. These shrewd guess are known as *hypotheses* (singular, *hypothesis*). It must strongly emphasized that a hypothesis is not a proved conclusio It cannot be given acceptance without satisfactory eviden in its favour. Scientists make many hypotheses and disca or change them when they are shown to be unsatisfactor Among the discarded hypotheses are the following: the ear is the centre of the universe; the earth is flat; heat is a flu germs can be produced by non-living material; the weight plants is mainly made up of substances derived from the so it is possible to construct a square of exactly the same ar as a circle; heavy bodies fall more quickly in a vacuum th light bodies.

All geometric conclusions made as a result of observati or measurement should be regarded as hypotheses requiri proof. The hypothesis that the bisectors of the angles

ilateral triangles are concurrent was quite reasonable, you found that it was too narrow and altered it to embrace ngles of all kinds. The new hypothesis may be regarded having a certain degree of probability on its side, but it st be proved before it can be accepted. You may be rested in learning that it can be proved, but the proof is a e too difficult at this stage.

### EXERCISES

1. Three times within a week, nine-year-old Bill was sent his mother to the same store, and each time his change was rrect. The first time, there were ten cents missing; the nd time, thirty; the third time, twenty. He was unable give an explanation. Form three hypotheses, each of ch accounts for the shortages. How would one seek firmation or disproof of each of the hypotheses?

In the following exercise, all the members of the class uld co-operate.

2. Each member of the class should draw a triangle $ABC$ which $AB = AC$. These isosceles triangles should be of ying sizes. Find the middle point $P$ of $BC$. From $P$, w perpendiculars $PD$ and $PE$ to $AB$ and $AC$ respectively. m $B$ and $C$, draw perpendiculars $BX$ and $CY$ to $AC$ and respectively. Make the indicated entries in a table on blackboard.

| me f pil | Length of $BX$ in cm. | Length of $CY$ in cm. | Length of $PD$ in cm. | Length of $PE$ in cm. | Sum of lengths of $PD$ and $PE$ in cm. |
|---|---|---|---|---|---|
| | | | | | |

State two hypotheses which may be drawn from the d obtained as the results of accurate construction and car measurement.

Let us consider first the hypothesis arising from the ent in columns 2 and 3. Can it be extended to non-isosc triangles?

Now let us turn to the hypothesis formed after examina of the figures in column 2 (or 3) and column 6. Can it extended to non-isosceles triangles?

Instead of using the middle point of the base $BC$, take point $P$ *anywhere* on $BC$ and draw the perpendiculars and $PE$ to $AB$ and $AC$. Is it necessary that $P$ should b the middle point of $BC$ in order that the agreement betw the figures in column 2 (or 3) and column 6 should be served?

Re-word the hypothesis to take into account the information obtained.

**3.** Some mathematicians have interested themselves trying to find a formula that will always yield prime numb (You will recall that a prime number has no factors ex itself and 1.)

Here is one of the formulas which have been tried:

$$x^2 + x + 41.$$

Substitute 0 for $x$. What number is obtained? I prime?

Substitute 1 for $x$. Did you obtain a prime num If a number of pupils co-operate, the formula can quickly tested for 2, 3, 4, 5, . . . . . 30.

Has any substitution yielded a number which is not pr Have you confidence in the formula? Would you say tha truth of the following generalization has been establis *All substitutions of positive integers for x in the expre*

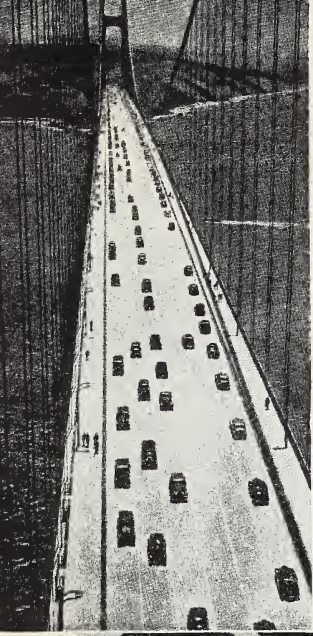
$+ x + 41$ *yield prime numbers.* Would you say that the owing hypothesis is justified? *It is probable that, whatever itive integer is substituted for* $x$ *in the expression* $x^2 + x + 41$, *result will be a prime number.*

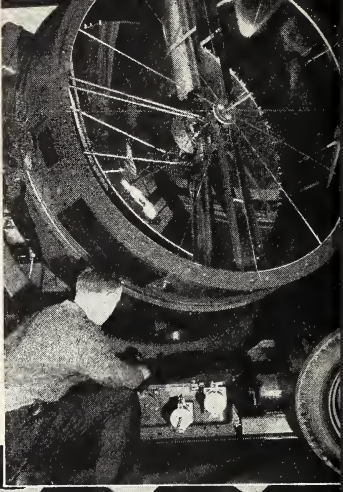
Now substitute the numbers 31 to 40 inclusive. Do the ılts agree with the generalization? Has the hypothesis n confirmed or disproved?

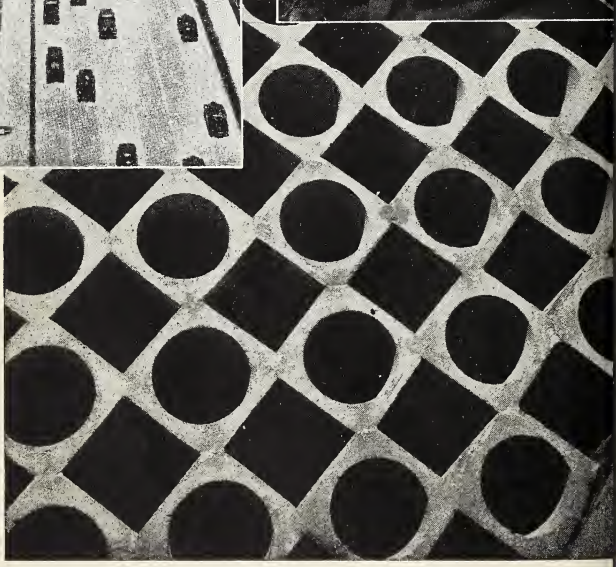
When 40 is substituted for $x$, the result is $1600 + 40 + 41$ 1681. Unfortunately (for both generalization and hyhesis), 1681 is not prime. It is the square of 41.

This exercise illustrates the need for caution in making eralizations. Forty instances agreed with the generalizaı; the forty-first contradicted it. Also, it illustrates ıirably a statement made many years ago by an eminent glish biologist.

> *"Every hypothesis is bound to explain, or, at any rate, ot to be inconsistent with, the whole of the facts it professes explain or account for; and if there is a single one of these acts which can be shown to be inconsistent with (I do not nerely mean inexplicable by, but contrary to) the hypothesis, he hypothesis falls to the ground—it is worth nothing. )ne fact with which it is positively inconsistent is worth as nuch, and is as powerful in negativing the hypothesis, as ive hundred."*
>
> T. H. Huxley—*Essays.*







*Top left: Parallel lines in San Francisco's Golden Gate Bridge.*
*(From Ewing Galloway, N*

*Top right: Circles, radii and sectors in a searchlight.*
*(National Film Board Ph*

*Bottom: This pattern of squares and circles belongs to a noise-s*
*pression medium built by the National Research Council for test*
*aircraft engines.* *(National Film Board Ph*

CHAPTER VII

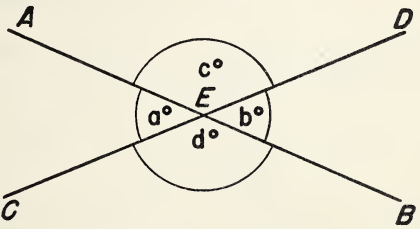
# THE EXPLORATION OF GEOMETRY IS CONTINUED

*"In a contemplative fashion, and a tranquil frame of mind,*
*Free from every kind of passion, some solution let us find."*

W. S. Gilbert

### PROPOSITION 10. THEOREM

***If two straight lines intersect, the vertically opposite angles are equal.***




***Given:*** $AB$ and $CD$ are two straight lines intersecting at $E$.

***Required:*** To prove that (1) $a = b$,

(2) $c = d$.

***Proof:***

$CED$ is a straight line, (Why?)

$\therefore a^\circ + c^\circ =$ a straight angle.

$AEB$ is a straight line, (Why?)

$\therefore c^\circ + b^\circ =$ a straight angle.

$\therefore a + c = c + b.$

From each of these equals take away $c$, and the remainders are equal. (Justification?)

That is, $a = b.$

Similarly, it may be proved that

$c = d.$

**Discussion of Proposition 10.** Let us consider three lin in the proof.

$$a^\circ + c^\circ = \text{a straight angle.}$$
$$c^\circ + b^\circ = \text{a straight angle.}$$
$$\therefore a + c = c + b.$$

There is nothing unreasonable in the drawing of the co clusion in the third line. The whole procedure does not diff essentially from the following argument:—

Phyllis is 63 inches in height.
Renaline is 63 inches in height.
Phyllis is the same height as Renaline.

The right to draw these two conclusions is assumed. T assumption is stated as follows:—

*Assumption 3*

***Things which are equal to the same thing are equal to one another.***

**EXERCISES**

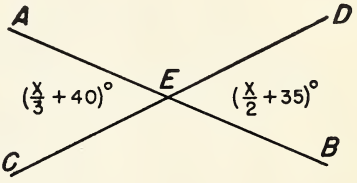
1. The last lines of the proof of Proposition 10 are follows: "Similarly, it may be proved that $c = d$." Pro that $c = d$.

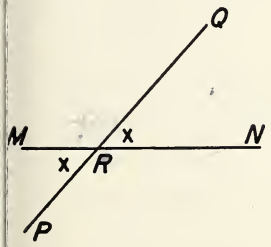
2. Two straight lines $AB$ and $CD$ intersect at $E$. If t angle $AEC$ is 15°, find the size of (*a*) $\angle BED$, (*b*) $\angle BE$ (*c*) $\angle AED$.

3. Two vertically opposite angles contain $3x°$ and $(x + 80)°$. What is the size of each of these two angles?

4. If two vertically opposite angles contain $8x°$ and $(x + 90)°$, what is the size of each?

5. Calculate the size (*a*) $\angle AEC$, (*b*) $AED$, (*c*) $\angle DEB$, (*d*) $\angle CEB$.



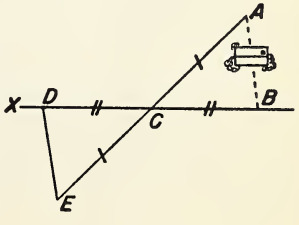




6. $MN$ is a straight line. Two straight lines $PR$ and $RQ$ are drawn so that $\angle MRP = \angle QRN$.

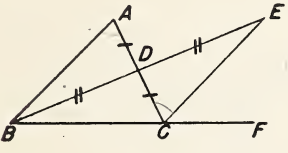
Prove that $PR$ and $RQ$ are in the same straight line. Which of the three numbered assumptions (p. 112, and p. 138) was used in your proof? What other assumption entered into your proof?

7. State a converse of Proposition 10. Can it be proved?

8. A man wished to find the distance from $A$ to $B$, but was unable to do so directly. He laid down a line $XB$ and marked on it a point $C$ from which both $A$ and $B$ were visible. Then he found a point $D$ in $XB$ such that $AC$ was equal to $CB$.



In the line $AC$ produced, he found a point $E$ such that $E$ was equal to $CA$. He measured $DE$ and took its length as th length of $AB$. Prove that his procedure was correct.




9. $D$ is the middle point $AC$. $BD$ is produced to $E$ mak ing $DE$ equal to $BD$. Prov that $\angle DCE = \angle DAB$.

10. If, in the figure of Exe cise 9, $BC$ is produced to $F$, prov that $ACF$ is greater than $\angle A$.

The angle $ACF$ is called an ***exterior angle*** of the triang $ABC$.

11. $H$ is the middle point of $BC$. $AH$ is produced to $K$ making $HK$ equal to $AH$. Prove that $\angle HCK = \angle HBA$.

12. If, in the figure of Exercise 11, $AC$ is produced to $G$, prove that $\angle BCG$ is greater than $\angle B$.

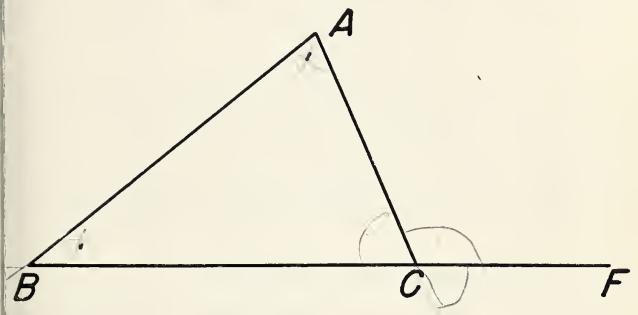
13. $BC$ in the figure of Exercise 11 is produced to $F$. Wh is the angle $BCG$ equal to the angle $ACF$?

Note that the result of Exercises 9 and 10 is a proof th the exterior angle $ACF$ of the triangle $ABC$ is greater than t angle $BAC$, which is one of the two interior opposite angles.

Note also that the result of the proofs of Exercises 11 a 12, combined with the fact brought out in Exercise 13, is a pr that the exterior angle $ACF$ of the triangle $ABC$ is grea than the angle $ABC$, which is one of the two interior oppos angles.

PROPOSITION 11. THEOREM

***If one side of a triangle is produced, the exterior ngle so formed is greater than either of the interior pposite angles.***




***Given:*** $ABC$ is a triangle in which $BC$ is produced to $F$.

***Required:*** To prove that $\angle ACF$ is greater than either $\angle BAC$ or $\angle ABC$.

***Proof:*** (A proof is unnecessary for all who have taken the ouble to do Exercises 9-13 in the preceding set.)

**Discussion of Proposition 11.** Recall your proof of Exercise 10 in the preceding set. (This, of course, is part of the roof of Proposition 11.) In your proof, you made use of an ssumption not previously encountered.

***Assumption 4***

***A whole is greater than its part.***

When you have proved that $\angle ACF$ is greater than $\angle BAC$ it is permissible to condense the remainder of the proof as follows:

If $AC$ is produced to $G$, it may be similarly proved that $\angle BCG$ is greater than $\angle ABC$.

But $\angle BCG = \angle ACF$ (Vertically opp. angles)

$\therefore \angle ACF$ is greater than $\angle ABC$.

* * *

Give a list of the previously proved propositions used in the proof of Proposition 11.

**Indirect proof.** Occasionally, in your study of geometry you will meet a method of proof which is frequently used in everyday life.

Let us consider the following proof:—

***Given:*** 1. A hit-and-run driver killed a pedestrian at the intersection of Criss and Cross Streets.

2. A garage man reported that marks on the fender of $X$'s car aroused his suspicions.

3. Two persons of good reputation testified that, at the time the accident occurred, $X$ was in a town one hundred and twenty miles away.

***Required:*** To prove that $X$ is not guilty.

***Proof:*** There are only two possibilities: (1) $X$ is guilty (2) $X$ is not guilty.

Let us suppose that $X$ is guilty.

Then he was present at the scene of the accident. But at the same moment, he was also a hundred and twenty miles away.

This is impossible.

Therefore, our supposition (of the guilt of $X$) is false.

Therefore, $X$ is not guilty.

We began with two contradictory possibilities. They ere the only possibilities. If one was true, the other was lse. Provisionally assuming the truth of one of the possi- ilities, we reasoned correctly and arrived at a manifestly rong conclusion. Hence, our assumption (the guilt of $X$) as false and its opposite true. A proof of this type is said o be ***indirect.***

In abbreviated form, indirect proofs are common.

In our next example, a mother proves indirectly that her vo-year-old child did not cross a stream.

Joyce (an older sister of the child Gustavus Adolphus): How did Gustavus Adolphus get back from the other side?"

Mother: "When did you see him on the other side?"

Joyce: "A few minutes ago."

Mother: "You must have mistaken the Bunnicomb child or him. Feel Gusty's stockings. They are as dry as uff."

We shall write the proof in full.

***Given:*** 1. A child was seen on the bank of a stream.

2. If the child was G. A., he must have walked through the allow water with his shoes and stockings on because he is too oung to untie his laces.

3. A few minutes after the child was seen, G. A.'s stockings ere dry.

***Required:*** To prove that the child was not G. A.

***Proof:*** Either G. A. walked across the stream or he did not.

Suppose that he did walk across.
Then his stockings are wet.
But they are not wet.
Therefore, he did not walk across the stream.

The indirect method is not confined to proofs where there re only two possibilities. If there are three possibilities, one

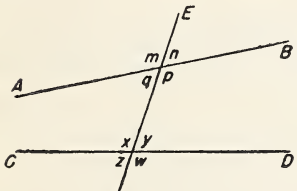
may be established by showing that two of them cannot occu

***It is not an uncommon error in the everyday use of indirect proof to overlook one or more possibilities.***

We shall consider an example of the use of indirect pro in geometry, but first it is necessary to add to your vocabular of geometric terms.

DEFINITION OF 'PARALLEL LINES': ***If two straight line in the same plane do not meet, however far they a produced in either direction, they are parallel.***

The symbol $\|$ means 'parallel' or 'is parallel to'.




In the figure, $EF$ is called ***transversal*** to $AB$ and $CD$.

The angles $m$, $n$, $z$, and $w$, a called ***exterior*** angles.

The angles $q$, $p$, $x$, and $y$, a called ***interior*** angles.

The angles $q$ and $y$ a ***alternate*** angles; $p$ and $x$ for another pair of alternate angle

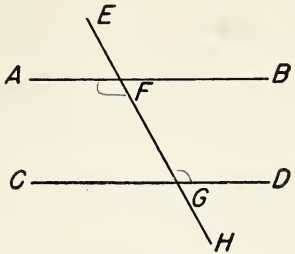
The angles $y$ and $n$ are called ***corresponding*** angle Other pairs of corresponding angles are as follows: $m$ and $w$ and $p$; $z$ and $q$.

### PROPOSITION 12. THEOREM

***If a transversal meets two straight lines, making th alternate angles equal, the two straight lines are paralle***

***Given:*** $EH$ is a transversal which meets two straight lin $AB$ and $CD$ at $F$ and $G$ respectively, making $\angle AFG$ equal t $\angle FGD$.

***Required:*** To prove that $AB \| CD$.




***Proof:*** There are two possibilities:—(i) $AB$ is parallel to ), (ii) $AB$ is not parallel to $CD$.

If $AB$ is not parallel to $CD$, then the two lines will meet en produced. Suppose that they meet at $P$.

Then $PFG$ is a triangle.

Now $\angle AFG = \angle FGD$. (Why?)

at is to say, an exterior angle of $\triangle PFG$ is equal to an erior opposite angle.

This is impossible. (Why?)

$\therefore$ the supposition that $AB$ and $CD$ meet when produced vards $B$ and $D$ is false.

Similarly, it can be shown that $AB$ and $CD$ do not meet en produced towards $A$ and $C$.

$$\therefore AB \parallel CD.$$

**Discussion of Proposition 12.** (Optional) Consider the owing assertion: *If Tom waited to shave, he was late for the lding.* By a logical process called ***contraposition,*** we are bled to draw the inference that *If Tom was not late for the lding, he did not wait to shave.* Each of the two statements aid to be the ***contrapositive*** of the other.

A few additional examples of contraposition will be given:

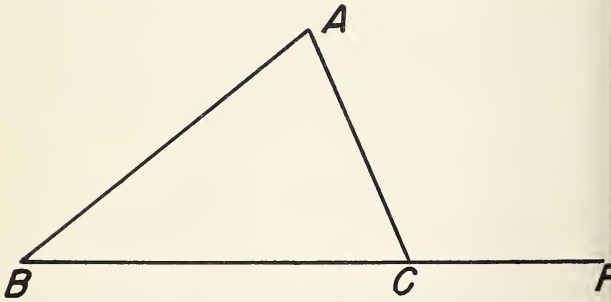
If the fathers are brothers, the boys are first cousins.
If the boys are not first cousins, the fathers are not brothers.

If Jim is under twenty-one, he is not entitled to vote.
If Jim is entitled to vote, he is not under twenty-one.

If the lines are not parallel, $\angle X$ is not equal to $\angle Y$.
If $\angle X$ is equal to $\angle Y$, the lines are parallel.

If $A$ is $B$, then $C$ is $D$.
If $C$ is not $D$, then $A$ is not $B$.

If $P$ is not $Q$, then $R$ is not $S$.
If $R$ is $S$, then $P$ is $Q$.




Let us look at the figure of Proposition 11. $BA$ and $F$ are not parallel because they meet at $B$. It was shov that the alternate angles $ACF$ and $A$ made by the tra versal $AC$ are not equal. In other words, it has be proved that ***if two straight lines are not parallel, t alternate angles are not equal.*** By contraposition, derive the conclusion of Proposition 12, namely, that ***if t alternate angles are equal, the two straight lines a parallel.***

***A corollary*** to a proposition is a theorem which follo immediately from that proposition, requiring no independe proof. It is clear that Proposition 12 is a corollary of P position 11. Logicians have criticized Euclid, and those w have followed his example, for giving an independent proof a corollary.

### PROPOSITION 13. THEOREM

***If a transversal meets two straight lines, making o corresponding angles equal, the two lines are rallel.***

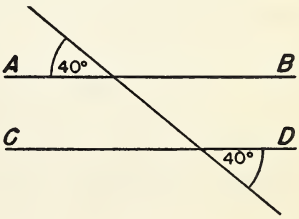
The proof is left as an exercise for the student.

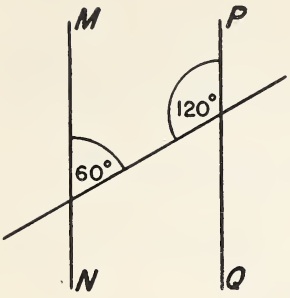
**Discussion of Proposition 13.** (Optional) Once more, us look at the figure of Proposition 11.

$BA$ and $AC$ are not parallel lines. It was proved that $ACF$ is not equal to $\angle B$. These two angles, it will be ted, are corresponding angles made by the transversal $BF$ h the lines $BA$ and $AC$. In brief, it has been proved that ***two straight lines are not parallel, two corresponding gles made by a transversal are unequal.*** By contra-sition, the following corollary is obtained: ***if a transversal eets two straight lines making two corresponding gles equal, the two lines are parallel.*** (Prop. 13.) It clear that independent proofs of both Propositions 12 and are superfluous.

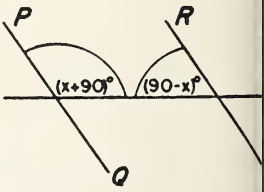
### EXERCISES

1. Prove $AB \parallel CD$.

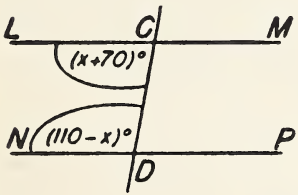





2. Prove $MN \parallel PQ$. Wha the sum of the two interior ang on the same side of the tra versal?




3. Prove $PQ \parallel RS$. What is the sum of the two interior angles on the same side of the transversal?




4. Prove $LM \parallel NP$. W is the sum of the angles $L$ and $CDN$?

If you can do Exercises 2, 3, and 4, you will have no di culty in giving a proof of Proposition 14.

### PROPOSITION 14. THEOREM

***If a transversal meets two straight lines, making t interior angles on the same side of the transver supplementary, the two straight lines are parallel.***

## EXERCISES

xercises 2-4 may be omitted by those who have not studied the optional paragraphs on p. 145 and p. 146).

1. Express in the 'If . . . then . . .' form each of the lowing statements:—

(*a*) All equilateral triangles are equiangular.

(*b*) All right angles are equal.

(*c*) All triangles with their sides respectively equal are ual in area.

(*d*) All circles of equal area have equal radii.

2. Give in 'If . . . then . . . ' form the contrapositive of ch statement in Exercise 1.

3. Give the contrapositive of each of the following state-nts:—

(*a*) If $x-p=0$, then $x=p$.

(*b*) If $a$ is unequal to $b$, then $a-b$ is unequal to zero.

(*c*) If $m$ is not equal to $n$, then $p$ is not equal to $q$.

(*d*) If $AO$ and $OB$ are in the same straight line, then $AOB$ a straight angle.

(*e*) If the liquid turns blue litmus red, then it contains an d.

(*f*) If Marion went to Normal School, then she sold her ycle.

4. Let us accept the statement, "If the rain had continued fall, the river would be in flood".

Are we arguing logically (that is, correctly) if we say, "But e river is not in flood; therefore, the rain did not continue to l"?

5. Prove that it is impossible to have more than one right gle in a triangle.

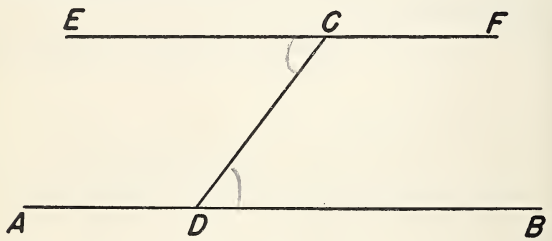
6. Two straight lines $PQ$ and $RS$ bisect each other. Prov
($a$) that $PR$ is parallel to $QS$, ($b$) that $PS$ is parallel to $QR$.

7. Prove that the sum of the angles of a triangle must b
less than three right angles.

8. $AB$ is a given straight line and $C$ is a given point outsid
$AB$. $C$ is joined to any point $D$ in $AB$. At the point $C$
an angle $DCE$ is made equal to the angle $CDB$ and alternat
to it. (p. 144). What is true of the lines $EC$ and $AB$? Why

## PROPOSITION 15. PROBLEM

***Through a given point, to draw a straight line paralle
to a given straight line.***




***Given:*** $C$ is a given point, and $AB$ is a given straight lin

***Required:*** To draw through $C$ a straight line $\parallel AB$.

***Construction:*** In $AB$ take any point $D$. Join $CD$. A
the point $C$ in $DC$, make $\angle DCE$ equal to $\angle CDB$ and alter
nate to it. (At given point in given straight line construc
$\angle$ equal to given $\angle$.)

Produce $EC$ to $F$. Then $EF$ is the required parallel.

***Proof:*** $EF$ and $AB$ are two straight lines and $CD$ is
transversal.

The alternate $\angle$s $ECD$ and $CDB$ are equal. (Why?)

$\therefore EF \parallel AB$.

## EXERCISES

1. To the best of your knowledge, describe each statement l also its converse as *true* or *false*.

(*a*) All numbers that end in 4 are exactly divisible by 2.

(*b*) If you are laughing, your mouth is open.

(*c*) All leap years have 366 days.

(*d*) All human beings require oxygen.

(*e*) All creatures with six legs are insects.

(*f*) All Scotch pines have needles in a cluster.

(*g*) If $x = a$, then $x - a = 0$.

(*h*) If $m$ is greater than $n$, then $n$ is less than $m$.

(*i*) All right angles are equal.

2. Give the converse of each of the following statements:—

(*a*) If a transversal meets two straight lines, making the ernate angles equal, the two straight lines are parallel.

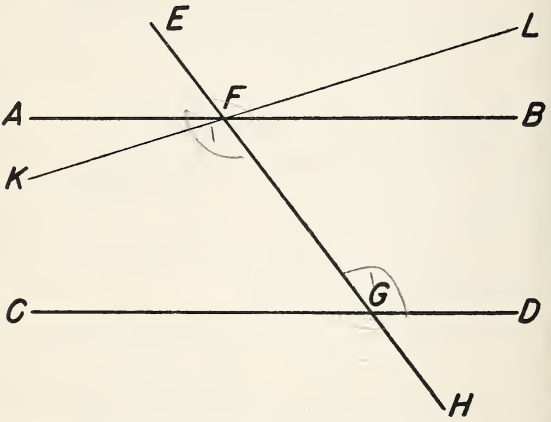
(*b*) If a transversal meets two straight lines, making two responding angles equal, the two straight lines are parallel.

(*c*) If a transversal meets two straight lines, making the erior angles on the same side of the transversal supplentary, the two straight lines are parallel.

## PROPOSITION 16. THEOREM 6.

***If a transversal meets two parallel straight lines, i makes the alternate angles equal.***

***Given:*** A transversal $EH$ meets the parallel straight lin $AB$ and $CD$ at $F$ and $G$ respectively.




***Required:*** To prove that $\angle AFG = \angle FGD$.

***Proof:*** Either $\angle AFG = \angle FGD$, or $\angle AFG$ is not equ to $\angle FGD$.

If $\angle AFG$ is not equal to $\angle FGD$, at $F$ make $\angle KFG$ equ to $\angle FGD$.

Produce $KF$ to $L$.
$\angle KFG = \angle FGD$. (Why?)
$\therefore KL \parallel CD$. (Why?)
But $AB$ also $\parallel CD$. (Why?)

That is, two intersecting straight lines are parallel to the sam straight line.

This is impossible. (Assumption)

$\therefore \angle AFG$ is not unequal to $\angle FGD$.
That is, $\angle AFG = \angle FGD$.

ıse of contraposition obtain a corollary of Proposition 16.

**Discussion of Proposition 16.** In this proposition you e met the first proof of a converse—the converse of Pro- tion 12. It provides us with our second geometric example ıe use of the indirect method of proof.

Consider the following statements in the proof of Proposi- 16:—

'That is, two intersecting straight lines are parallel to the e straight line. This is impossible." We have written word 'Assumption' in the margin. Attention to this mption was directed by the mathematician Playfair, and ıame is associated with it.

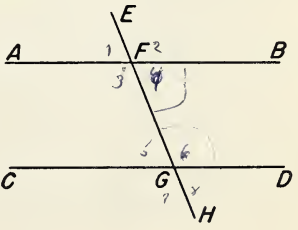
***Playfair's Axiom***

***Assumption 5***

***Two straight lines which intersect one another cannot be both parallel to the same straight line.***

**EXERCISES**

. $AB$ is parallel to $CD$ and is a transversal. Prove that ıngle $EFB$ is equal to the esponding angle $FGD$.




3 + 4 = 180°

2. In the figure of Ex. 1, show that the angles $BFG$ and $FG$ are supplementary.

The two exercises in the preceding set are of sufficie importance to be given the status of propositions. T general enunciations will be given.

### PROPOSITION 17. THEOREM

***If a transversal meets two parallel straight lines, makes corresponding angles equal.***

### PROPOSITION 18. THEOREM

***If a transversal meets two parallel straight lines, makes the two interior angles on the same side togeth equal to two right angles.***

**A backward glance.** Under what conditions are t straight lines parallel?

1. They are parallel if two alternate angles made by transversal are equal. (Proposition 12)

2. They are parallel if two corresponding angles made a transversal are equal. (Proposition 13)

3. They are parallel if two interior angles on the sa side of a transversal are supplementary. (Proposition 14)

What properties are true of angles made when a transver meets two parallel straight lines?

1. The alternate angles are equal. (Proposition 16)

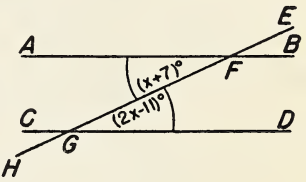
2. Corresponding angles are equal. (Proposition 17)

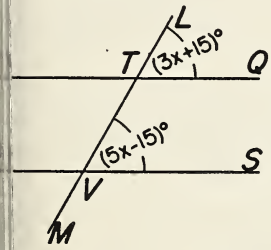
3. The interior angles on the same side of the transversal supplementary. (Proposition 18)

It will be noted that (*a*) Proposition 16 is the converse of position 12, (*b*) Proposition 17 is the converse of Proposition and (*c*) Proposition 18 is the converse of Proposition 14.

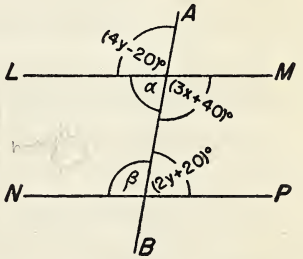
## EXERCISES

1. *AB* and *CD* are parallel *EH* is a transversal. Calate the size of each of the owing angles: (*a*) *EFB*; *HGD*.



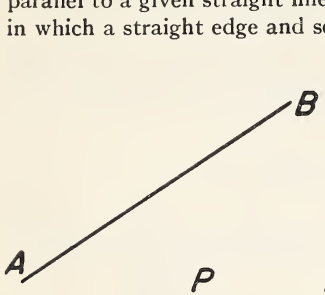

2. *PQ* and *RS* are parallel and *LM* is a transversal. Calculate the size of each of the following angles: (*a*) *PTV*; (*b*) *RVT*.

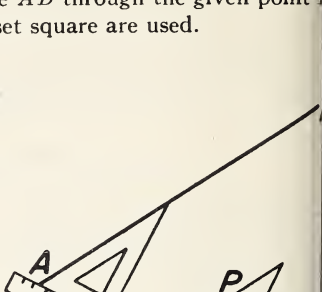



3. *LM* and *NP* are parallel *AB* is a transversal. Calate the size of each of the owing angles: (*a*) α; (*b*) β.

4. Of what proposition is the following assertion a corollary? *ight lines which are perpendicular to the same straight line are llel to each other.*

5. Give a proof for the following method of drawing parallel to a given straight line $AB$ through the given point in which a straight edge and set square are used.




6. $ABC$ is a triangle in which $AB = AC$. $DE$ is a li drawn parallel to $BC$ and cutting $AB$ and $AC$ in the poi $D$ and $E$ respectively. Show that $\angle ADE = \angle AED$.

7. *If two straight lines are parallel to the same straight li prove that they are parallel to each other.*

DEFINITION: ***A parallelogram is a four-sided figure which the opposite sides are parallel.***

The symbol for 'parallelogram' is || gm; for 'parallelogra || gms.

8. *Prove that the opposite angles of a parallelogram are eq* (The conclusion of this exercise may be used without proof succeeding exercises.)

9. *If the diagonals of a quadrilateral bisect each other, sh that the quadrilateral is a parallelogram.*

10. *In a quadrilateral $ABCD$, $AB$ is equal and parallel to Show that $ABCD$ is a parallelogram.* (In Euclid, this cise was a proposition. The conclusion may be used iout proof in succeeding exercises.)

11. (*a*) *If the opposite sides of a quadrilateral are equal, show the quadrilateral is a parallelogram.*

(*b*) (i) Give the converse of the statement in (*a*). (ii) s the converse require proof? (Do not try to prove the verse.)

(*c*) (i) Give the contrapositive of the statement in (*a*). Does the contrapositive require proof?

12. If one angle of a parallelogram is a right angle, prove the remaining angles are right angles.

Definition: ***A rectangle is a parallelogram in which one le is a right angle.*** Why do we not say that a rectangle parallelogram containing four right angles?

13. Construct a rectangle of length 2.3 inches and width nches. Measure the diagonals.

14. Without using a protractor, construct a parallelogram $CD$ from each of the following sets of data:—

(*a*) $AB = 2.1$ in., $BC = 1.5$ in., $\angle B = 135°$.
(*b*) $AB = 2.4$ in., $BC = 1.8$ in., $\angle B = 67\frac{1}{2}°$.
(*c*) $BC = 4.1$ cm., $CD = 4.1$ cm., $\angle C = 112\frac{1}{2}°$.

15. $ABC$ is a triangle in which the angle $B$ is a right angle the middle point of $AC$. The line $BD$ is drawn and duced to $E$, making $DE$ equal to $BD$. Prove that $ABCE$ rectangle.

16. *ABC is a triangle in which BC is produced to D. is parallel to BA. (a) Prove that* $\angle ACD = \angle A + \angle$ *(b) Prove that the sum of the interior angles of the triangle is* 1

**We recall and add to our knowledge about definiti** You may remember (p. 97) that we have mentioned characteristics of a good definition:—

1. Preferably it should deal with properties possessed the thing defined, not with properties it lacks. (The defini of a lazy person as one who is not industrious is unsatisfacto

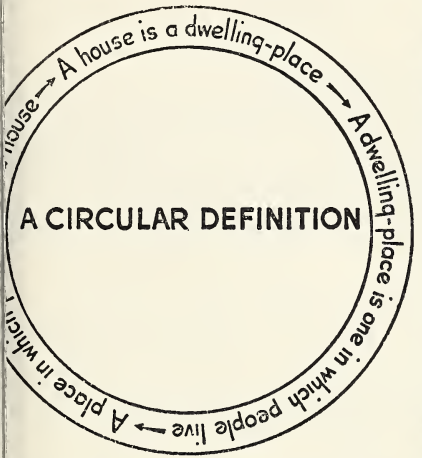
2. It is reversible. (To define a cabbage as a veget would be foolish; the definition is too wide because not vegetables are cabbages. Another example of a defini that is too wide is the following: ***A rectangle is a parall gram.*** If we define a reptile as a snake, our definition is narrow for there are reptiles which are not snakes, for insta the turtle. When a definition is reversible, it is neither wide nor too narrow.)

Now we shall refer to three additional characteristics.

3. It contains nothing superfluous. (We have def a rectangle as a parallelogram in which one angle is a r angle (p. 157). If we had said that a rectangle is a para ogram containing four right angles, our definition would h been superfluous to the extent of three right angles, bec it can be proved (Ex. 12, p. 157) that, if a parallelogram tains one right angle, then the remaining three angles be right angles).

4. It is not circular. (The 'circular' or 'teeter-to definition contains openly or in concealment the term w is being defined. Many years ago, a judge gave a jury following helpful 'definition' of *murder*: "Murder, gentle is when a man is murderously killed. It is the murd

constitutes murder in the eyes of the law. Murder in t is —— murder." Other illustrations of the attempt efine a word in terms of itself are the following: *a gardener is one who looks after a garden*; *deafness is the state of being deaf*; *a solution is a solute dissolved in a solvent.* Because we have many words of Latin and Greek origin, the circularity of some definitions is not always apparent at first glance. If we say that *Life is the operation of vital forces*, we are really trying to raise our-




es by means of our shoe straps, because 'vital' means ting to life', (Latin: *vita*, life). Similarly unsatisfactory are e: *light is that which is emitted by luminous bodies* (Latin: *n*, light); *a half-drunk man is one who is semi-intoxicated.*)

. It is not expressed obscurely or figuratively. (Consider definition: *an isosceles triangle is one that has equal sides.* we mean by 'sides' the two lines which are not horizontal he paper? With this meaning, the definition is incorrect rding to present-day usage. Do we mean by 'sides' the e sides? If we do, the definition is too narrow, because xcludes triangles with only two equal sides. Since the e of a definition is in the avoidance of vagueness, we should ne an isosceles triangle as one that contains ***at least two*** l sides. The following figurative statements are unsatis-

factory definitions: *the camel is the ship of the desert*; *the mo is the queen of night.*)

## EXERCISES

Criticize the following definitions:

1. A rectangle is a parallelogram.

2. An acute-angled triangle is one that contains an acu angle.

3. Perpendicularity is a property of two perpendicul lines.

4. A horse is a beast of burden.

5. A conservative person is one who favours conservatis

6. The Milky Way is the path of souls.

7. A glutton is one who is intemperate at the table and e to excess.

8. A motion is a resolution that has been moved.

9. Rudeness is a characteristic of those who lack go manners.

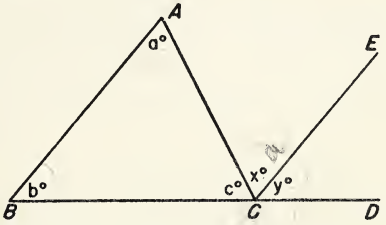
CHAPTER VIII

# N WHICH WE ADVANCE ANOTHER MILE

## PROPOSITION 19. THEOREM

***If one side of a triangle is produced, the exterior gle so formed is equal to the sum of the two interior, posite angles.***

***The sum of the three interior angles of a triangle is ual to two right angles.***




***Given:*** $ABC$ is a triangle in which $BC$ is produced to $D$.

***Required:*** To prove that (1) $\angle ACD = \angle A + \angle B$;
(2) $\angle A + \angle B + \angle C = 180°$.

***Proof:*** Draw $CE \parallel BA$.

(1) $CE \parallel BA$, and $AC$ is a transversal,
$\therefore x = a.$ (Why?)
$CE \parallel BA$, and $BD$ is a transversal,
$\therefore y = b.$ (Why?)
$\therefore x + y = a + b.$ (Justification?)
That is to say, $\angle ACD = \angle A + \angle B$.

(2) $a + b = x + y.$ (Why?)
$\therefore a + b + c = c + x + y.$ (Justification?)
$= 180.$ (Why?)
That is to say, $\angle A + \angle B + \angle C = 180°$.

**Discussion of Proposition 19.** The property of triangl just proved, namely, that the sum of the interior angles two right angles, is extremely important as well as interestin It was known to Greek mathematicians twenty-five centuri ago.

Some of you may wonder why we first proved that exterior angle of a triangle is greater than either of t interior, opposite angles, and then proved that the exteri angle is equal to their sum. If you can give a satisfacto reason, you have gained insight into the system in whi propositions are arranged.

## EXERCISES

1. *If two angles of one triangle are respectively equal to t angles of another triangle, the third angle of the one is equal to third angle of the other.*

2. The sum of the acute angles in a right triangle is right angle.

3. If one angle of a triangle is equal to the sum of other two, the triangle is right-angled.

4. From a point outside a line, only one perpendicular be drawn to the line.

5. By drawing a parallel to $BC$ through the point $A$, ove that the sum of the interior angles of the triangle $ABC$ two right angles.

6. By joining $A$ to any point in $BC$, prove that the sum of interior angles of the triangle $ABC$ is two right angles.

7. Show that each angle of an equilateral triangle is 60°.

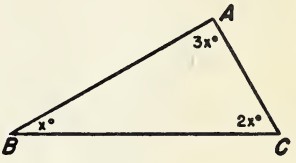
8. Using only straight edge and compasses, construct gles of 60°, 120°, 30°, 75°, and 105°.

9. Using only straight edge and compasses trisect an gle of (*a*) 90°, (*b*) 45°.

10. Prove that each acute angle in a right-angled isosceles angle is 45°.

11. In an isosceles triangle $ABC$, each of the angles $B$ and $C$ double the angle $A$. Find the size of each angle.

12. Calculate the size of each gle in the triangle $ABC$.



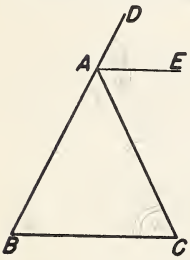

13. In a triangle, one angle is uble another. Their sum is 0°. Find the size of each of e three angles of the triangle.

14. In a triangle $ABC$, $AB = AC$. Show (*a*) that the rpendicular from $A$ to $BC$ bisects the angle $A$, (*b*) that the rpendicular bisects $BC$.

15. One of the acute angles of a right triangle is three nes the other. How many degrees does each contain?

16. Any point $D$ is taken in the triangle $ABC$. Prov that the angle $BDC$ is greater than the angle $A$.

17. From the middle point $C$ of the straight line $AB$, $C$ is drawn in any direction outside $AB$. If $CD = CA = CE$ prove that the angle $ADB$ is a right angle.

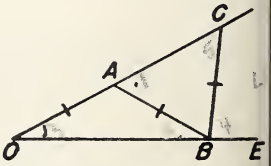


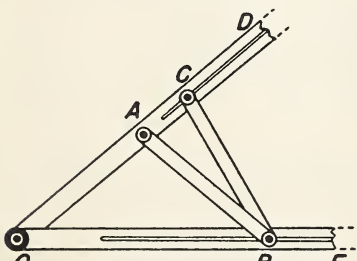

18. Prove that at least tw angles of any triangle are acut

19. In the triangle $ABC$ $AB = AC$. $AE$ is the bisecto of the exterior angle $CAD$. Prov that $AE$ is parallel to $BC$.

20. State and prove a converse of Exercise 19.

21. In the figure, $OA = AB = BC$. $OB$ is produced to $E$. Prove that the angle $CBE$ is three times the size of the angle $AOB$.

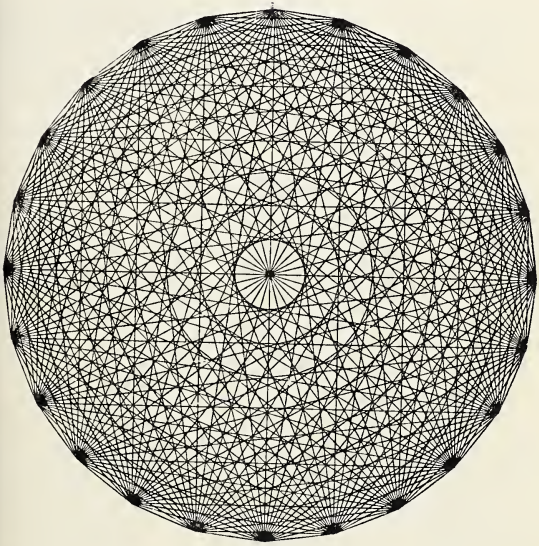



A device for the tr section of angles is base on the figure for Exercis 21. Four thin strips cardboard are cut as ind cated in the accompany ing sketch. The device jointed at $O$, $A$ and $B$ slides in a slot in $OE$; slides in a slot in $OD$. Co struct one of these device and find out how to use i

**We meet some new members of the polygon family.** You recall that the triangle is a polygon of three sides, and t the quadrilateral is a polygon of four sides. A few other ygons will be mentioned.

| | |
|---|---|
| Pentagon . . . . . | 5 sides |
| Hexagon . . . . . | 6 sides |
| Octagon . . . . . | 8 sides |
| Decagon . . . . . | 10 sides |
| Dodecagon . . . . | 12 sides |
| Quindecagon . . . . | 15 sides |



Twenty-four-sided polygon with all diagonals

*oduced from "Multi-sensory Aids in the Teaching of Mathematics", ind permission of the National Council of Teachers of Mathematics and the Bureau of Publications, Columbia University, N.Y.*

## EXERCISES

Find, without measurement, the number of right angl in the sum of the interior angles of (*a*) a quadrilateral, (*b*) pentagon, (*c*) a hexagon, (*d*) an octagon.

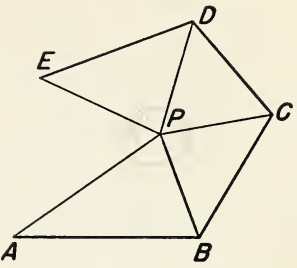
In your work-book make the following table:—

| *Name of polygon* | *Number of sides* | *Number of right angles in the sum of the interior angles* |
|---|---|---|
| *Triangle*<br>*Quadrilateral*<br>*Pentagon*<br>*Hexagon*<br>*Octagon* | | |

In the table, write the required information. Compa the number of right angles and the number of sides. Try discover a formula which will enable you to calculate t number of right angles from the number of sides. If y obtain a formula, find if it 'works' with a decagon and wit dodecagon. If a polygon has 60 sides, how many right ang are there in the sum of the interior angles? If a polygon ha sides, what is the number of right angles?

* * *

Consider a poly- $ABCDE\ldots$ of $n$ s.




Take any point $P$ nin the polygon join it to each ex.

Since there are $n$ ngles and the sum he angles in each ngle is 2 right ang- the sum of the angles of all the triangles $= 2n$ right angles.

But this number includes the sum of the angles at $P$, 4 t angles.

Therefore, the sum of the angles $A$, $B$, $C$, $D$, $E$, etc. $=$ $-4)$ right angles.

## EXERCISES

**1.** State the number of right angles in the sum of the rior angles of the following figures:—(*a*) a 7-sided polygon; a 20-sided polygon; (*c*) a polygon of 100 sides.

**2.** Produce in succession the sides of a pentagon. What he sum of the interior angle and the exterior angle at each ex? What is the total sum for the five vertices? Sub- t the sum of the interior angles. What is the sum of the rior angles?

**3.** Find the sum of the exterior angles of (*a*) a triangle, (*b*) adrilateral, (*c*) a hexagon, (*d*) an $n$-sided polygon.

**4.** *If both pairs of opposite angles of a quadrilateral are l, the figure is a parallelogram.*

(*Hints*: Let the numbers of degrees in the angles be $x$, $y$, and $y$. What is the sum of these four angles? What rence concerning two adjacent angles can be drawn? To t conclusion does this inference lead you?)

## *Hints on Doing Exercises*

For some pupils, the doing of geometrical exercises presen difficulty. If the following advice is taken, there will cor gradually a sense of mastery accompanied by self-confiden and, "believe it or not", pleasure.

1. You should have a thorough knowledge of the co clusions of the propositions and of the important exercises yo have met. This does not involve a wonderful feat of memor it does involve systematic review.

2. Distinguish very clearly between what you are giv and what you are required to prove. Make a note of ther and keep them constantly in mind.

3. Draw a reasonably accurate and fairly large figur (If the exercise involves an accurate construction, 'reasonabl accuracy is, of course, unsatisfactory!) Do not draw particul figures; they may lead you astray. For example, if you a given a triangle, do not draw an isosceles triangle or a rig triangle. If you are given a parallelogram, do not draw rectangle or a parallelogram with four equal sides.

4. Mark clearly on the figure what you are given.

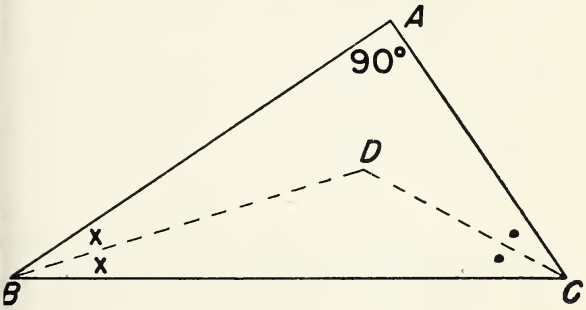
5. Mobilize the conclusions of propositions which ma have a bearing on the exercise. For example, if you have prove that two angles are equal, you summon to your aid the propositions you know dealing with the equality of angl —the congruency propositions, the equal angles in an isoscel triangle, vertically opposite angles, the equal angles made b a transversal and parallel lines, etc. Then you ask yours which of these is most likely to be of help. Should I join tho points to obtain a triangle? Should I produce that line?

6. Be dogged. Do not give up easily. *Accept help on as a last resort.* Watching other people do deductions w never give you any degree of expertness.

7. Be flexible. If one method of attack does not bring ults, try another.

8. Understand the method of analysis and employ it ely. We shall give some examples of its use.

**The use of analysis in doing geometrical exercises.** Let consider the following deduction:—




***Given:*** $ABC$ is a triangle in which $\angle A = 90°$. $BD$ is bisector of $\angle ABC$. $CD$ is the bisector of $\angle ACB$. bserve that we have marked in the figure what is given.)

***Required:*** To prove that $\angle BDC = 135°$.

**The analysis.** If $\angle BDC$ is 135°, what conclusion can I aw?

The sum of the $\angle$s $DBC$ and $DCB$ is $180° - 135°$, i.e., 45°.

If $\angle DBC + \angle DCB = 45°$, what conclusion can I draw?

The sum of the $\angle$s $ABC$ and $ACB$ is 90°. (If the sum of half angles is 45°, the sum of the whole angles must be 90°.)

If $\angle ABC + \angle ACB = 90°$, what conclusion can I draw?

The $\angle A$ is $180° - 90°$, i.e., 90°.

Hurrah! I know that $\angle A$ is 90°. Therefore, I can ve the deduction.

Now we shall write the analytic proof and the synthetic (or ordinary) proof side by side for the sake of comparison.

| *Analytic proof* ('*Backward*' *Method*) | *Synthetic proof* ('*Forward*' *Method*) |
|---|---|
| 1. $\angle BDC = 135°$, if $\angle DBC + \angle DCB = 45°$. | 1. $\angle BAC = 90°$. (Given |
| 2. $\angle DBC + \angle DCB = 45°$, if $\angle ABC + \angle ACB = 90°$. | 2. $\therefore \angle ABC + \angle ACB = 90$ |
| 3. $\angle ABC + \angle ACB = 90°$, if $\angle A = 90°$. But $\angle A = 90°$. (Given) | 3. $\therefore \angle DBC + \angle DCB = 45$ |
| 4. $\therefore \angle BDC = 135°$. | 4. $\therefore \angle BDC = 135°$. |

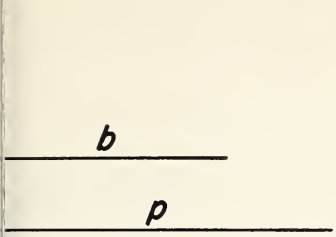
Note that the synthetic proof (of which all our proofs of propositions are examples) begins with what is given and ends with the required conclusion; the analytic proof begins with the required conclusion and ends with what is given. It is customary to end the analysis (which is usually done mentally or as 'rough work') with the line preceding Step 4, and then to proceed with the synthetic proof. Because of the presence of pitfalls in the wording of the analysis, *it is recommended that the 'backward' procedure be used as a means of discovering the synthetic proof*, rather than as a method of proof.

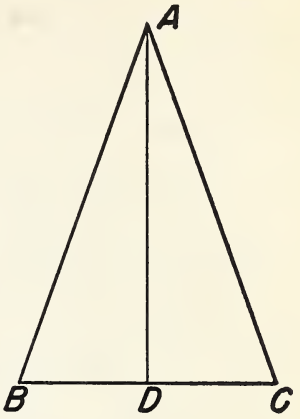
* * *

A second illustration of analysis will be given.

Construct a triangle $ABC$ in which $AB = AC$, the side $BC$ is equal to a given line $b$, and the perpendicular from $A$ to $BC$ is equal to another given line $p$.

Let us imagine that we have solved the problem and that the following figure is correct.




We shall examine the figure and try to discover possibly ful properties. We recall that, if $\triangle ABC$ is isosceles, the pendicular from $A$ to $BC$ bisects $BC$. (Ex. 14, p. 163).

This gives us the key to the solution.

Draw a line and cut off $BC$ equal to the given line $b$.

Bisect $BC$ at $D$. At $D$, erect a perpendicular and cut off $DA$ equal to $p$.

Join $AB$ and $AC$.

Then $ABC$ is the required triangle.

***Proof:*** In $\triangle$s $ABD$, $ACD$,

$BD = DC$, (Why?)

$AD$ is common,

$\angle ADB = \angle ADC$, (Why?)

$\therefore \triangle ABD \equiv \triangle ACD$. (Why?)

$\therefore AB = AC$,

$BC = b$, (Why?)

$AD = p$. (Why?)

$\therefore ABC$ is the required triangle.

In general, the analytic and synthetic procedures are a follows:—

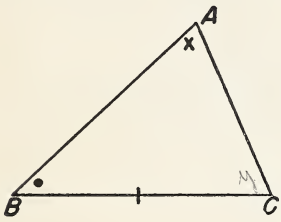
***Given:*** $P$ is true.

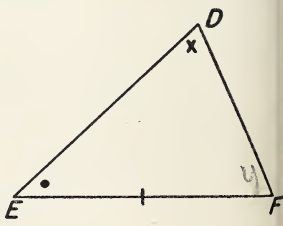
***Required:*** To prove that $S$ is true.

| *Analytic proof* | *Synthetic proof* |
|---|---|
| 1. $S$ is true if $R$ is true. | 1. $P$ is true (Given) |
| 2. $R$ is true if $Q$ is true. | 2. $\therefore Q$ is true. |
| 3. $Q$ is true if $P$ is true. | 3. $\therefore R$ is true. |
| But $P$ is true. (Given) | 4. $\therefore S$ is true. |
| 4. $\therefore S$ is true. | |

### PROPOSITION 20. THEOREM

***If two angles and a side of one triangle are respectivel equal to two angles and the corresponding side of anothe triangle, the triangles are congruent.***




***Given:*** $ABC$ and $DEF$ are triangles in which $\angle A = \angle D$, $\angle B = \angle E$, and $BC = EF$.

***Required:*** To prove that $\triangle ABC \equiv \triangle DEF$.

***Proof:*** $\angle A = \angle D$ and $\angle B = \angle E$,
$\therefore \angle C = \angle F$. (Why?)

Apply $\triangle ABC$ to $\triangle DEF$ so that $B$ falls on $E$ and so that $BC$ lies along $EF$.

$BC = EF.$ (Why?)

$\therefore C$ falls on $F$.

$\angle B = \angle E.$ (Why?)

$\therefore BA$ lies along $ED$ or $ED$ produced.

$\angle C = \angle F.$ (Why?)

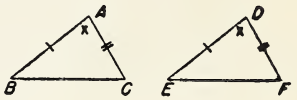
$\therefore CA$ lies along $FD$ or $FD$ produced.

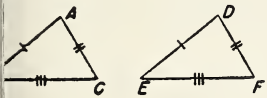
$\therefore A$ must fall on $D$, the only point common to $ED$ d $FD$.

$\therefore \triangle ABC$ coincides with $\triangle DEF$ and is congruent th it.

**Summing up our knowledge of congruent triangles.** We all now bring together the three propositions concerning the ngruency of triangles with which we have dealt.

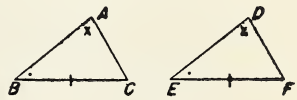
1. If $AB = DE$,
$AC = DF$,
d incl. $\angle BAC =$ incl. $\angle EDF$,
then $\triangle ABC \equiv \triangle DEF$.
(Prop. 1)



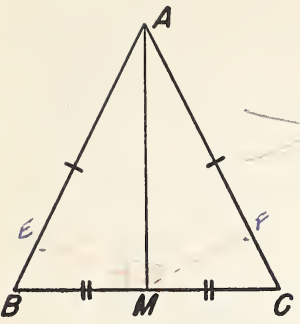

2. If $AB = DE$,
$AC = DF$,
and $BC = EF$,
then $\triangle ABC \equiv \triangle DEF$.
(Prop. 3)

3. If $BC = EF$,
$\angle BAC = \angle EDF$,
and $\angle ABC = \angle DEF$,
then $\triangle ABC \equiv \triangle DEF$.
(Prop. 20)




## EXERCISES

1. Construct a triangle $ABC$ in which $BC = 4.9$ cm., $A = 45°$, and $\angle B = 75°$. (Do not use your protractor.)




2. In the triangle $ABC$, i $AB = AC$ and $BM = MC$ show that $AMB$ is a righ angle.

3. Prove that two righ triangles are congruent if th hypotenuse and an acute angl in one are equal to the hypo tenuse and an acute angle i the other.

4. In the triangle $ABC$, $AB = AC$. The perpendicula from $A$ to $BC$ meets it at $D$. Prove ($a$) that $AD$ bisects $BC$ ($b$) that $AD$ bisects $\angle BAC$.

5. $ABC$ is a triangle. $M$ is the middle point of $BC$. ($a$ What name is given to the line $AM$? ($b$) Prove that th perpendicular from $B$ to $AM$ equals the perpendicular fron $C$ to $AM$.

6. In the triangle $ABC$, $AB = AC$ and $M$ is the middl point of $BC$. Show that the perpendiculars from $M$ to $A$ and $AC$ are equal.

7. In the triangle $ABC$, $AB = AC$, the perpendicula from $B$ to $AC$ meets $AC$ at $D$, and the perpendicular from to $AB$ meets $AB$ at $E$. Prove that $BD$ is equal to $CE$.

NOTE: *The conclusions of exercises 8, 9, and 10 may be use without proof in succeeding exercises.*

***8. Prove that each diagonal of a parallelogram divides it into two congruent triangles.***

***9. Prove that the opposite sides of a parallelogram are equal.***

***10. Show that the diagonals of a parallelogram bisect each other.***

11. Any point is taken on the bisector of an angle. Prove at the perpendiculars from this point to the arms of the gle are equal.

12. Prove that the diagonals of a rectangle are equal.

13. The diagonals $AC$ and $BD$ of a rectangle $ABCD$ inter-ct at $O$. Show that $OA = OD = OC = OB$.

14. *In a right triangle, the straight line drawn from the right gle to the midpoint of the hypotenuse is equal to half the hypot-use.* (In a right triangle, the side opposite the right angle is lled the ***hypotenuse.***)

15. *Prove that the straight line joining the midpoints of two les of a triangle is parallel to the third side and equal to half of* (*Hints*: Let $ABC$ be the triangle. $D$ is the midpoint of $B$, and $E$ the midpoint of $AC$. Join $DE$. Produce $DE$ to so that $EF = DE$. Join $CF$. Compare $\triangle$s $ADE$, $CFE$. hat kind of figure is the quadrilateral $BCFD$? (See Ex. 10, ge 157.)

EFINITION: ***A square is a rectangle with two adjacent des equal.*** Recall the definition of a parallelogram and of rectangle.

Why do we not define a square as a rectangle with four ual sides? . . . Note that a parallelogram is a special type quadrilateral, a rectangle is a special type of parallelogram, d a square is a special type of rectangle.

Give a definition of a square beginning as follows: A square a parallelogram in which . . . . .

16. *Using straight edge and compasses, construct a square on given straight line. Prove that the figure is a square.*

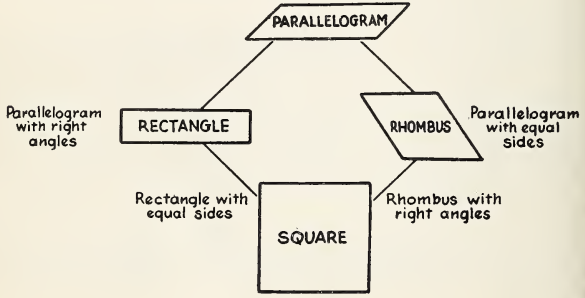
17. Prove that a diagonal of a square bisects the angles rough which it passes. What is the size of each half angle?

**18.** Prove (*a*) that the diagonals of a square are equal, (*b* that they bisect each other at right angles.

**19.** Given a diagonal of a square, construct the square.

DEFINITION: ***A rhombus is a parallelogram with tw adjacent sides equal.*** Are the four sides equal? Note tha a square is a special kind of rhombus.

The following arrangement brings out clearly the relation among the four figures. It will be noted that the mos special type of parallelogram is the square.




**20.** Using straight edge and compasses, construct a rhom bus of side 2 inches and containing an angle of 45°.

**21.** Prove that a diagonal of a rhombus bisects the angl through which it passes. Is this also a property of a rectangl

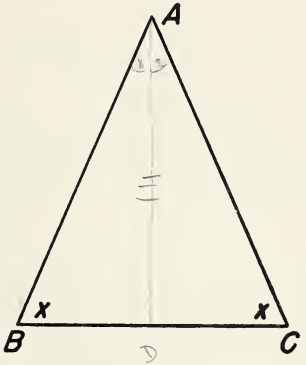
**22.** Prove that the diagonals of a rhombus bisect eac other at right angles.

**23.** In what special types of quadrilateral with the opposit sides parallel are the diagonals equal?

**24.** Prove that, in an equilateral triangle, the perpendicula drawn from the vertices to the opposite sides are equal.

### PROPOSITION 21. THEOREM

***If two angles of a triangle are equal, the sides oppo- these angles are equal.***




***Given:*** $ABC$ is a triangle in which $\angle B = \angle C$.

***Required:*** To prove that $AB = AC$.

***Proof:*** (*Hint*: Bisect $\angle BAC$.)

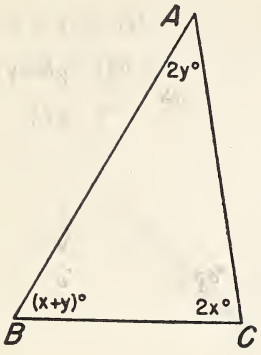
### EXERCISES

1. Of what proposition is Proposition 21 the converse?

2. If the three angles of a triangle are equal, the triangle quilateral. Prove:

3. (*a*) $DB$ and $DC$ are the bisectors of the angles $ABC$ and $B$ of a triangle in which $AB$ is equal to $AC$. Show that $C$ is an isosceles triangle.

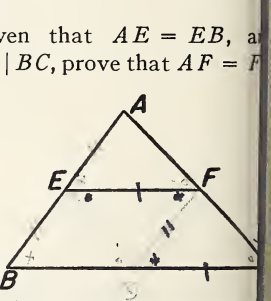
(*b*) Show that $AD$ bisects the angle $BAC$.




4. (*a*) Calculate the size of $\angle$
(*b*) If $(x-y)° = 20°$, calculate t
size of each of $\angle$s $A$ and $C$.

5. In the quadrilateral $ABC$
$AB = AD$, and $\angle ADC = \angle AB$
Show (*a*) that $CB = CD$; (*b*) th
$AC$ bisects the angles through whi
it passes.

6. Given that $AE = EB$, a
that $EF \parallel BC$, prove that $AF = F$




7. Prove that, in the figure of Exercise 6, (*a*) $EF = \frac{1}{2}BC$; (*b*) $\triangle AEF = \frac{1}{4} \triangle ABC$.

8. In the figure of Exercise 6, show that $EF$ bisects all li
drawn from $A$ to $BC$.

9. A page of foolscap is ruled with lines so that consecut
lines are the same distance apart. A straight line is ru
obliquely on the page. Show that the segments formed on t
line by the rulings are equal.

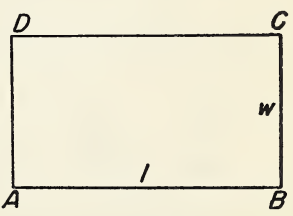
CHAPTER IX

## CONCERNING AREA

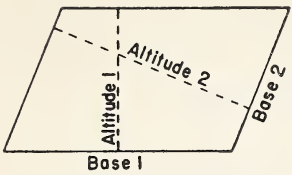
We shall next consider a few propositions dealing with *a*, that is to say with the *quantity of surface within a ed line.* It is quite true that we have met the topic pre- usly in this book, because congruent triangles are of equal a. We know that two triangles whose corresponding sides l angles are equal are of equal area; we know also from eral knowledge that the converse is not necessarily true. angles (and other figures) may be equal in area, but their pes may be quite different. When we write "$\triangle ABC \equiv \triangle DEF$", we mean that the two triangles are equal in all pects, but when we write "$\triangle ABC = \triangle DEF$", we assert y that the areas are equal.

You have learned that the number of square units in the a of a rectangle is equal to the product of the numbers of ts of length in its length and width. (The width is the ne as the altitude.) The formula used, you will recall, is $= l\,w$.

If the rectangle is a square, e formula becomes $A = l^2$.

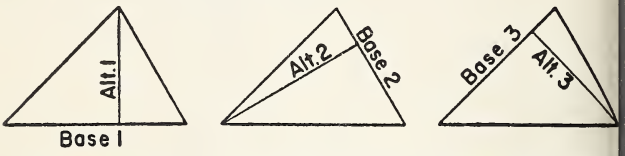



Frequently, we say that the ea of a rectangle $ABCD$ is $B \cdot BC$. When we do so, $AB$ ke $l$) must be understood to be a number of units of length, d $BC$ (like $w$) also a number of units of length.




The altitude of a parallelogram is the perpendicular distance between two opposite sides. Thus, every parallelogram has two altitudes.

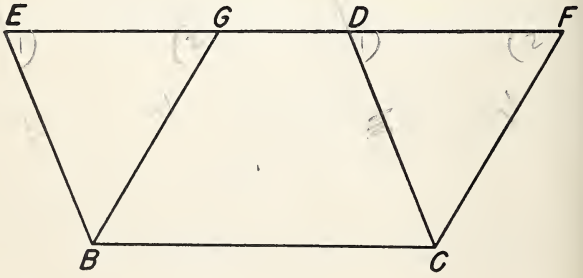
Any side of a parallelogram may be regarded as its base. If we use Base 1, we use along with it Altitude 1. Similarly Base 2 and Altitude 2 go together.




The altitude of a triangle is the perpendicular distance from a vertex to the base. Since any side may be used as base, we have three bases each with its corresponding altitude.

### PROPOSITION 22. THEOREM

***If two parallelograms are on the same base and between the same parallels, they are equal in area.***



***Given:*** $BCDE$ and $BCFG$ are || gms. on the same base $BC$ and between the same parallel lines $BC$ and $EF$.

***Required:*** To prove that || gm. $BCDE$ = || gm. $BCFG$.

***Proof:***

$BCDE$ is a || gm.,

$\therefore BE = CD.$ (Why?)

$BE \parallel CD.$

$\therefore \angle GEB = \angle FDC.$ (Why?)

Similarly, $\angle EGB = \angle DFC.$

In $\triangle$s $EBG$ and $DCF$,

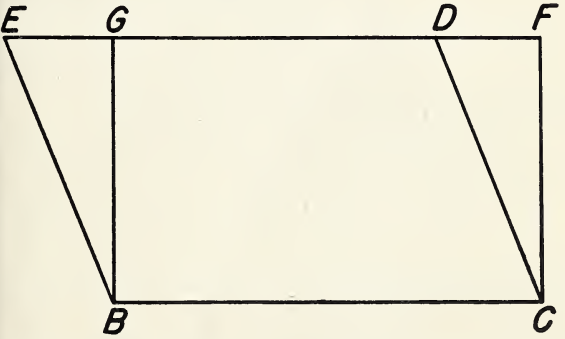
$$\begin{cases} BE = CD, \\ \angle GEB = \angle FDC, \\ \angle EGB = \angle DFC. \end{cases}$$

$\therefore \triangle EBG = \triangle DCF.$ (Why?)

From the figure $BCFE$, subtract in turn the equal $\triangle$s $DCF$ and $EBG$.

$\therefore$ || gm. $BCDE$ = || gm. $BCFG$. (Why?)

**Discussion of Proposition 22.**




If the general parallelogram $BCFG$ had been a rectangle, would have obtained the conclusion:

|| gm. $BCDE$ = rect. $BCFG$.

But the area of the rectangle $BCFG$ is $BC.CF$ (or the produ of the number of units of length in base and altitude). Bu $CF$ is the altitude of || gm. $BCDE$. Hence, if $A$ = no. o units of area in a parallelogram, if $b$ = no. of units of length i base, and if $h$ = no. of units of length in altitude, we arriv at the following formula:—

***Area of parallelogram***

$$A = bh$$

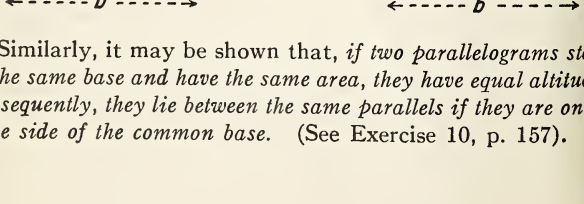
* * *

Consider two parallelograms. The measure of the area one is $A_1$, that of the other, $A_2$. They have equal bases an equal altitudes.

$$A_1 = bh,$$
$$A_2 = bh.$$
$$\therefore A_1 = A_2.$$

That is, the areas are equal.

Thus we see that *if two parallelograms stand on equal bas and have the same altitude, they are equal in area.*




Similarly, it may be shown that, *if two parallelograms sta on the same base and have the same area, they have equal altitude Consequently, they lie between the same parallels if they are on t same side of the common base.* (See Exercise 10, p. 157).

## EXERCISES

1. A square has a 3-inch side. Give (*a*) the perimeter, (*b*) area.

2. A rectangle is 5 cm. long and 4 cm. wide. State (*a*) the imeter, (*b*) the area.

3. The area of a square is 36 square inches. What is perimeter?

4. The length of a rectangle is 6 centimetres and its area 4 square centimetres. What is the width of the rectangle?

5. The width of a rectangle is 5 inches and its area is 30 are inches. Calculate the perimeter.

6. Calculate the area of a parallelogram the base and tude of which are respectively 8 cm. and 5 cm.

7. Give the altitude of a parallelogram with a base of 7 hes and an area of 28 square inches.

8. Find the area of each square the side of which (measured nits of length) is given below:—
(*a*) 31; (*b*) $7n$; (*c*) $2\frac{1}{2}p$; (*d*) $0.5r$; (*e*) $0.02s$;
(*f*) $x + 1$; (*g*) $a + b$; (*h*) $a - 5$; (*i*) $x + 7$; (*j*) $x - 11$.

9. The areas in square units of a number of squares are en below. Find the number of units of length in the side ach square:—(*a*) $9p^2$; (*b*) $49r^2$; (*c*) $\frac{1}{4}a^2$; (*d*) $x^2/9$; (*e*) $0.04b^2$; $0.81q^2$; (*g*) $a^2 + 2a + 1$; (*h*) $x^2 - 2x + 1$; (*i*) $p^2 + 6p + 9$; $m^2 + 8m + 16$; (*k*) $n^2 - 14n + 49$.

10. The length and width in units of length are given for number of rectangles. Find the area of each rectangle:
(*a*) $3a$, $2b$; (*b*) $5a$, $4a$; (*c*) 4, $a + 2$; (*d*) $m + 3$, 6; (*e*) $a$, $+ 5$; (*f*) $m - 7$, $m$; (*g*) $3k$, $k + 3$; (*h*) $a + 2$, $a - 2$; (*i*) $+ y$, $x - y$; (*j*) $x + 5$, $x + 4$; (*k*) $m + 8$, $m + 8$; (*l*) $p + 6$, $+ 7$.

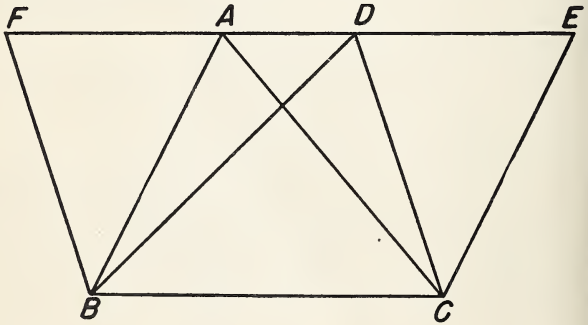
**11.** Give the altitudes of the rectangles whose length (measured in units of length) and areas (measured in square units) are respectively as follows:— (*a*) $3a$, $15a^2$; (*b*) $2x$, $8x$; (*c*) $4q$, $20q^2$; (*d*) $5a$, $25ab$; (*e*) $3c$, $3c^2 + 12c$; (*f*) $2x$, $8x - 10x$; (*g*) $c + d$, $c^2 + cd$.

**12.** The length of the base and the altitude (measured in units of length) are given for a number of parallelograms. Find the area of each.

(*a*) $2x$, $5y$; (*b*) $4x$, $3x$; (*c*) $a$, $a - b$; (*d*) $4x$, $2x - 6$; (*e*) $a + 2b$, $2a - b$; (*f*) $x + 2y$, $3x - 4y$.

## PROPOSITION 23. THEOREM

***If two triangles are on the same base and between the same parallels, they are equal in area.***




***Given:*** $ABC$ and $DBC$ are triangles on the same base $B$ and between the same parallel lines $BC$ and $AD$.

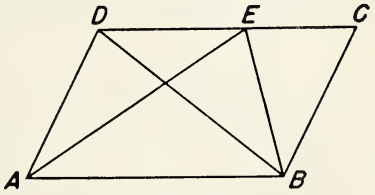
***Required:*** To prove that $\triangle ABC = \triangle DBC$.

***Proof:*** Draw $BF \parallel CD$, and $CE \parallel BA$, meeting $AD$ or $AD$ produced at $F$ and $E$ respectively.

$FD \parallel BC$, (Why?)
$FB \parallel DC$. (Why?)
$\therefore BCDF$ is a $\parallel$ gm.
Similarly, $BCEA$ is a $\parallel$ gm.
$\triangle ABC = \frac{1}{2} \parallel$ gm. $BCEA$, (Why?)
$= \frac{1}{2} \parallel$ gm. $BCDF$, (Why?)
$= \triangle DBC$. (Why?)

## PROPOSITION 24. THEOREM

***If a parallelogram and a triangle are on the same e and between the same parallels, the area of the ngle is half that of the parallelogram.***



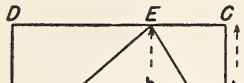

***Given:*** $ABCD$ is a $\parallel$ gm. and $EAB$ is a $\triangle$ on the same $AB$ and between the same parallel lines $AB$ and $DC$.

***Required:*** To prove that $\triangle EAB = \frac{1}{2} \parallel$ gm. $ABCD$.

***Proof:*** Join $DB$.

$\triangle EAB = \triangle DAB$, (Why?)
$= \frac{1}{2} \parallel$ gm. $ABCD$. (Why?)

OLLARY: ***The area of a triangle is one-half the area rectangle on the same base and between the same allels.***




If the base of the rectangle[1] co tains $b$, and its altitude $h$, units length, then the measure of its ar is $bh$.

Since a triangle which is betwe the same parallels as a rectangle h the same altitude, then the measure ($A$) of the area of triangle $= \frac{1}{2}\,bh$.

***Area of triangle***

$$A = \tfrac{1}{2}bh$$

* * *

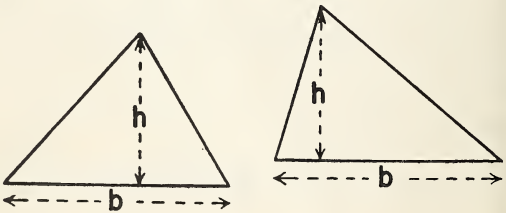
Consider two triangles with equal bases and altitudes.

Let the measure of the area of one be $A_1$, that of the oth $A_2$.

$$A_1 = \tfrac{1}{2}\,bh.$$
$$A_2 = \tfrac{1}{2}\,bh.$$
$$\therefore\ A_1 = A_2.$$

That is, the areas are equal.

Thus we obtain the result that, *if two triangles stand equal bases and have equal altitudes, they are equal in a*




[1]The length of a line in a diagram may be indicated by writing besid line either its length or the number that gives the measure of its length. example, if a line is $b$ feet long, label it either with *b feet* or, simply, with number $b$.

ilarly, it may be shown that *if two triangles of equal area id on the same base, they have equal altitudes. Consequently, have a common vertex or lie between the same parallels if they on the same side of the common base.* (See Ex. 10, p. 157).

### EXERCISES

1. The base and altitude of a triangle are respectively cm. and 6.5 cm. (*a*) Calculate the area. (*b*) Calculate area of a parallelogram on the same base and with the e altitude.

2. The area of a triangle is 3.6 sq. in. The length of the e is 1.8 in. (*a*) Find the altitude. (*b*) What would be the tude of a parallelogram of the same area and standing on same base?

3. The base and altitude of each of a number of triangles given below in units of length. Find the area of each. $3x$, $2a$; (*b*) $4x$, $5x$; (*c*) $2a$, $a + 1$; (*d*) $6x$, $x + 3$; (*e*) $c$, $c + d$; $c + 3$, $2x + 2$.

4. The base (in units of length) and the area (in units of .) are respectively given below for a number of triangles. d the altitude of each triangle. (*a*) $b$, $ab$; (*b*) $x$, $x^2/2$; $c$, $c^2 + 5c$; (*d*) $d + 3$, $d^2 + 3d$; (*e*) $x + y$, $xy + y^2$.

5. In a $\triangle ABC$, $BC = 4$ inches, $CA = 3$ inches, and the th of the perpendicular from $A$ to $BC$ is 2 inches. Find length of the perpendicular from $B$ to $AC$.

6. In a $\triangle ABC$, $BC = 3x$ units, $AC = 3$ units, the pendicular from $B$ to $AC = 4p$ units, and the perpendicular n $A$ to $BC = p$ units. Find (*a*) the value of $x$, and hence he length of $BC$.

7. In a parallelogram $ABCD$, $AB = 4x$ units, $BC = 2$ units, the altitude corresponding to base $AB = y$ units, an the altitude corresponding to base $BC = 8$ units. Find (a $x$, (b) the length of $AB$.

8. Prove that a median divides a triangle into two triangl of equal area.

9. Construct a parallelogram having two adjacent sid 2 inches and 3 inches and the contained angle 30°. Find i area.

10. Construct a parallelogram in which one side is 5.8 cm a diagonal is 9.1 cm., and the angle included between the giv side and diagonal is 45°. Find its area.

11. Construct a parallelogram $ABCD$ the area of which 30 sq. cm., and in which $AB$ is 6 cm., and the angle $DAB$ 60°. What is the length of $DA$?

12. The side $AB$ of a parallelogram $ABCD$ is 3 inches length. The area of the parallelogram is 6 square inch Draw one of the two lines on which $CD$ must lie. How ma parallelograms fulfilling the given conditions could be draw

13. Construct a triangle $ABC$ in which $BC = 6.3$ c $\angle A = 83°$, and $\angle B = 41°$. Find the area of the triang

14. (a) Construct a triangle $ABC$ the area of which is square centimetres, and in which $BC$ is 5 centimetres long a the angle $B$ is 75°. (Do not use your protractor.)

(b) Construct a triangle $DBC$ equal in area to the trian $ABC$ and in which the angle $DBC$ is 90°. Measure $DB$.

15. (a) A triangle $ABC$ has an area of 5 square inch $BC$ is 2 inches in length. On what line must the vertex lie?

(b) How many triangles fulfilling the given conditi could be drawn?

(*c*) On the line $BC$, as base, construct an isosceles triangle ıal in area to the triangle $ABC$.

**16.** (*a*) Construct a triangle $ABC$ in which $AB = 4.8$ ., $BC = 6.3$ cm., and $CA = 3.7$ cm.

(*b*) On the line $AC$ as base construct a rectangle with twice area of the triangle $ABC$.

(*c*) Find the area of (i) the rectangle, (ii) the triangle.

**17.** (*a*) On a line 4.6 cm. in length construct an equilateral ıngle.

(*b*) On one of the sides of the triangle construct a rectangle ıal in area to the triangle.

(*c*) Find the area of the triangle.

**18.** A rhombus has an area of 20 square centimetres. Each e is 5 centimetres in length. Construct the rhombus and asure the longer diagonal.

**19.** Show algebraically that, if two rectangles have equal es and equal areas, they have equal altitudes.

**20.** Two rectangles are equal in area. The altitude and e of one are respectively 4 in. and $5\frac{1}{2}$ in. If the altitude of other is 6 in., calculate the length of its base.

**21.** Show algebraically that, if two triangles have equal es and equal areas, they have equal altitudes.

**22.** The area of the triangle $ABC$ is three times that of the ngle $DEF$. The altitude and base of the triangle $ABC$ are nd 4 units respectively. $EF$ is 2 units. Find the length he perpendicular from $D$ to $EF$.

**23.** On the same base with a given parallelogram, construct arallelogram of the same area as the given parallelogram having an angle equal to a given angle.

**24.** On the same base with a given parallelogram, construc a parallelogram of the same area as the given parallelogra and having a second side equal to a given straight line. Und what conditions is this construction impossible?

**25.** On the same base with a given triangle, construct triangle of the same area as the given triangle, and having angle equal to a given angle.

**26.** On the same base with a given triangle, construct triangle of the same area as the given triangle, and having second side equal to a given straight line. Under wh conditions is this construction impossible?

**27.** Construct a rhombus equal in area to a given parallel gram.

**28.** The perimeter of a rectangle is 32 centimetres and base is 10 centimetres. Find its area.

**29.** The perimeter of a rectangle is 40 rods. The leng is 4 rods longer than the width. Find the area.

**30.** The perimeter of an isosceles triangle is 14 centimetr One of the equal sides is 5 centimetres. Find the area of t triangle.

**31.** Through each vertex of a triangle a line is dra parallel to the opposite side.

(*a*) How many equal parallelograms are formed?

(*b*) How many equal triangles are formed?

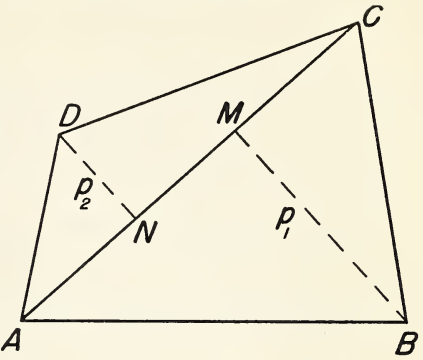
**32.** $ABCD$ is a parallelogram in which $AB = 5.9$ c $AD = 5.1$ cm., and $\angle DAB = 60°$. Construct two rectang equal in area to $ABCD$ and on unequal bases.

## The Area of a Quadrilateral

Let $ABCD$ be a adrilateral in ich $AC$, one of diagonals is wn and lies inside quadrilateral.

$DN \perp AC$. $BM$ $AC$.

Let $AC$, $BM$ and $N$ contain respec-ely $d$, $p_1$ and $p_2$ ts of length.




$$\begin{aligned}\text{Quad. } ABCD &= \triangle ABC + \triangle ADC \\ &= \tfrac{1}{2}dp_1 + \tfrac{1}{2}dp_2 \\ &= \tfrac{1}{2}d\,(p_1 + p_2).\end{aligned}$$

(*a*) If the quadrilateral $ABCD$ is such that $M$ and $N$ ncide, say at the point $L$, what inference can be drawn cerning the lines $BL$ and $DL$? * * * How is straight line $BLD$ described in relation to the quadrilateral? der these circumstances, what form would be taken by the nula for the area of $ABCD$?

(*b*) What would the formula become if $ABCD$ is kite-ped?

(*c*) If $AC$ and $DB$ bisect each other at right angles, what ne is given to the quadrilateral $ABCD$? Write the formula the area of this figure. What does it become when $= DB$?

(*d*) What is the formula for the area when the point $D$ is de the triangle $ABC$?

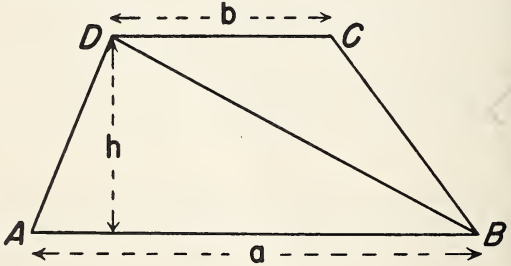
**33.** In a quadrilateral $ABCD$, the diagonal $BD = 10$ cm., perpendicular from $C$ to $BD = 8$ cm., and the perpen-

dicular from $A$ to $BD = 5$ cm. Calculate the area of th quadrilateral ($a$) if $A$ and $C$ are on the same side of $BD$, ( if they are on opposite sides.

**34.** Construct a quadrilateral $ABCD$ in which $AB = 5$ cm., $BC = 6.3$ cm., $CD = 6.6$ cm., $DA = 4.4$ cm., $\angle B = 83$ Find the area of the quadrilateral ($a$) by using the diagon $BD$, ($b$) by using the diagonal $AC$. (You may use the protra tor.)

**35.** You have seen that the measure of the area of a qua rilateral $ABCD$ is given by the following formula:—Qua $ABCD = \frac{1}{2}d\ (p_1 + p_2)$. ($a$) If the measure of the area $p_1^2 - p_2^2$, find the measure of $d$. ($b$) If $d = 2(p_1 +$ find the number of square units in the area. ($c$) If $p_1 = p_2$ $d/2$, find the measure of the area in terms of $d$. ($d$) Sh that the area of a square is half the area of the square on diagonal.

DEFINITION: ***A trapezium is a quadrilateral with a p of parallel sides.***




In the trapezium $ABCD$, $a$ and $b$ are the numbers of u of length in the parallel sides $AB$ and $CD$ respectively. number of units in the altitude is $h$. Join $DB$.

Number of units of area in $\triangle ABD = \frac{1}{2}ah$.
Number of units of area in $\triangle DCB = \frac{1}{2}bh$.

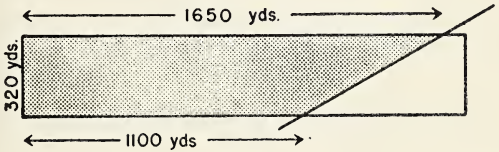
Hence, number of units of area in trapezium $ABCD$

$$= \tfrac{1}{2}\,ah + \tfrac{1}{2}\,bh,$$
$$= \tfrac{1}{2}\,(a + b)\,h.$$

That is, *to find the measure of the area of a trapezium, take the sum of the measures of the parallel sides and multiply by measure of the distance between them.*

36. A lawn has the form of a trapezium. The parallel s are 60 feet and 100 feet in length, and the distance ween them is 90 feet. Find in square feet the area of the n.

37. A farm in the shape of a rectangle is traversed by a h situated as shown in the sketch. Find in acres the area he shaded portion.




38. The formula for the area of a trapezium is $= \tfrac{1}{2}(a + b)\,h$. (*a*) To what does the formula reduce if 0? What figure is suggested by the new formula?

(*b*) If $b = a$, what does the formula become? What re is suggested?

## CHAPTER X

# RATIO AND PROPORTION

Suppose that you wish to compare two numbers such as 6 an 2. Two simple methods are as follows:

(1) You may say that 6 is 4 *more* than 2. This is comparis by ***subtraction.***

(2) You may say that 6 is 3 times 2, or that 2 is $\frac{2}{6}$ or of 6. This is comparison by ***division.***

When we compare two numbers by dividing one by the oth we say that we have found the ***ratio*** of the two numbers.

*Examples:* The ratio of the two numbers 8 and 12 is $\frac{8}{12}$ or thus 8 is $\frac{2}{3}$ of 12. The ratio of 25 to 20 is $\frac{25}{20}$ or $\frac{5}{4}$.

> *The ratio of the number* $a$ *to the number* $b$ *is* $\frac{a}{b}$.

Two quantities may be compared in a similar way. F example, the ratio of \$4 to \$20 is $\frac{4}{20}$ or $\frac{1}{5}$; the ratio of 4¢ to \$20 $\frac{4}{2000}$ or $\frac{1}{500}$; the ratio of 1 year to 1 century is $\frac{1}{100}$.

> *The ratio of two quantities of the same kind is the ratio of their measures when the quantities are expressed in the same units.*

### EXERCISES

1. What is the ratio of (*a*) 5¢ to 1 dollar; (*b*) 1 dollar to 5 ts; (*c*) 2 feet to 4 yards; (*d*) 4 yards to 2 feet; (*e*) 44 hours 8 hours; (*f*) 8 gallons to 2 quarts; (*g*) 9 square inches to 1 are foot; (*h*) 484 square yards to 1 acre; (*i*) $87\frac{1}{2}$ square s to 100 square rods; (*j*) 45° to 1 straight angle; (*k*) 30° to ght angle?

2. If $\triangle ABC$ has a base of 6 inches and an altitude of 4 es, and $\triangle DEF$ has a base of 8 inches and an altitude of 9 es, what is the ratio of the area of $\triangle ABC$ to the area of $DEF$?

**Proportion.** In the previous exercise you found that the o of the area of $\triangle ABC$ to the area of $\triangle DEF$ is the o of 1 to 3. Expressed briefly, this becomes

$$\frac{\triangle ABC}{\triangle DEF} = \frac{1}{3}.$$

INITION: ***If two ratios are equal, the four numbers in proportion.***

n the example of proportion used, namely, $\frac{\triangle ABC}{\triangle DEF} = \frac{1}{3}$, four magnitudes are $\triangle ABC$ (i.e., area of $\triangle ABC$), $DEF$, 1 and 3. The statement $\frac{\triangle ABC}{\triangle DEF} = \frac{1}{3}$ is read as ws: "The area of the triangle $ABC$ is to the area of the ngle $DEF$ as 1 is to 3".

Let us consider the following statement of proportion:—

$$\frac{a}{b} = \frac{c}{d}.$$

Multiply each fraction by $bd$ and we have

$$\frac{abd}{b} = \frac{cbd}{d}, \quad \text{or} \quad ad = bc.$$

f we now divide each of these equal expressions by $cd$, we :

$$\frac{ad}{cd} = \frac{bc}{cd}, \quad \text{or} \quad \frac{a}{c} = \frac{b}{d}.$$

To sum up, if $a$, $b$, $c$, and $d$ are in proportion then we ma write immediately

$$1.\quad \frac{a}{b} = \frac{c}{d},$$

or

$$2.\quad \frac{a}{c} = \frac{b}{d},$$

or

$$3.\quad ad = bc.$$

**EXERCISES**

1. If $\frac{a}{b} = \frac{c}{d}$, and $b = d$, is $a$ equal to $c$? Why?

2. If $\frac{a}{b} = \frac{c}{d}$, and $a = c$, is $b$ equal to $d$? Why?

3. If $\frac{AE}{EC} = \frac{AE}{EF}$, is $EC$ equal to $EF$? Why?

4. If $\frac{a}{b} = \frac{c}{d}$, is $\frac{b}{a}$ equal to $\frac{d}{c}$? Why?

5. If $\frac{a}{b} = \frac{c}{d}$, is $\frac{c}{a}$ equal to $\frac{d}{b}$? Why?

6. If $\frac{x}{3} = \frac{5}{6}$, find $x$.

7. If $\frac{2}{p} = \frac{3}{36}$, find $p$.

8. If $\frac{4}{15} = \frac{r}{30}$, find $r$.

9. If $\frac{9}{2} = \frac{27}{s}$, find $s$.

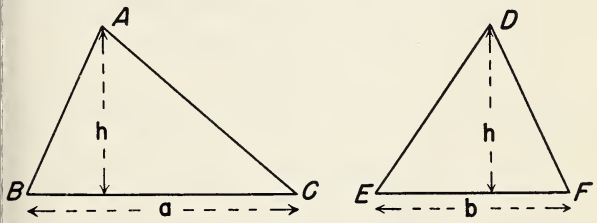
10. Find the ratio of $x$ to $y$ if

(a) $14x = 7y$;
(b) $7x = 14y$;
(c) $x = 7y$;
(d) $7x = y$;
(e) $px = qy$,
(f) $(m + n)x = ay$;
(g) $px = y(q + r)$;
(h) $\frac{x + y}{x - y} = \frac{6}{5}$.

11. If $\frac{a}{b} = \frac{b}{c}$, show that $b^2 = ac$.

## PROPOSITION 25. THEOREM

***If two triangles have equal altitudes, their areas are one another as their bases.***




***Given:*** $ABC$ and $DEF$ are triangles of equal altitude.

***Required:*** To prove that $\frac{\triangle ABC}{\triangle DEF} = \frac{BC}{EF}$.

***Proof:*** Let the altitude of each triangle contain $h$ units of length. Let $BC$ and $EF$ contain respectively $a$ and $b$ units of length.

$\triangle ABC = \frac{1}{2}ah.$ (Why?)
$\triangle DEF = \frac{1}{2}bh.$
$\therefore \frac{\triangle ABC}{\triangle DEF} = \frac{a}{b}.$ (Justification?)

**Discussion of Proposition 25.** In this proof we have us an assumption not encountered previously in a proposition.

> ***Assumption 6***
> ***If equals are divided by equals, the quotients are equal.***

You have used this assumption frequently in algebra work, and also the sister assumption given below.

> ***Assumption 7***
> ***If equals are multiplied by equals, the products are equal.***

**EXERCISES**

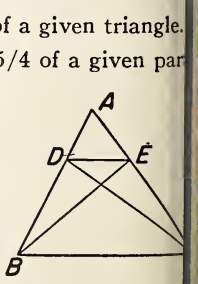
1. Triangles on equal bases are to each other as th altitudes.

2. If two parallelograms have equal altitudes, their are are to one another as their bases.

3. In the $\triangle ABC$, a point $D$ is taken in $BC$ such th $BD = \frac{1}{3} BC$. Show that $\triangle ADC = 2 \triangle ABD$.

4. Construct a triangle equal to 3/2 of a given triangle.

5. Construct a parallelogram equal to 5/4 of a given par lelogram.




6. In the $\triangle ABC$, $DE \parallel BC$.

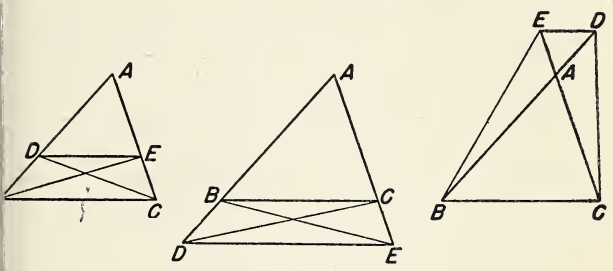
(*a*) Prove that $\triangle BDE = \triangle CDE$.

(*b*) Prove that $\dfrac{\triangle BDE}{\triangle ADE} = \dfrac{BD}{DA}$

7. Solve Ex. 6 when $D$ and $E$ fall on $AB$ and $AC$ produc

## PROPOSITION 26. THEOREM

***If a straight line is drawn parallel to one side of a ingle, it cuts the other two sides, or the other two es produced, proportionally.***




***Given:*** In $\triangle ABC$, $DE \parallel BC$.

***Required:*** To prove that $\frac{AD}{DB} = \frac{AE}{EC}$.

***Proof:*** Join $BE$ and $DC$.

$DE \parallel BC$. (Why?)

$\therefore \triangle BDE = \triangle CDE$. (Why?)

$\therefore \frac{\triangle ADE}{\triangle BDE} = \frac{\triangle ADE}{\triangle CDE}$. (Why?)

But $\frac{\triangle ADE}{\triangle BDE} = \frac{AD}{DB}$, (Why?)

and $\frac{\triangle ADE}{\triangle CDE} = \frac{AE}{EC}$, (Why?)

$\therefore \frac{AD}{DB} = \frac{AE}{EC}$.

**Discussion of Proposition 26.** If, in each of the thr figures, $D$ is placed on $AB$ and $E$ on $AC$, the proof applies all.

* * *

You will recall that if $\frac{a}{b} = \frac{c}{d}$, then $\frac{b}{a} = \frac{d}{c}$.

Similarly, since $\frac{AD}{DB} = \frac{AE}{EC}$, $\therefore \frac{DB}{AD} = \frac{EC}{AE}$.

Also, since $\frac{AD}{DB} = \frac{AE}{EC}$,

therefore $\frac{AD}{DB} + 1 = \frac{AE}{EC} + 1$.

That is, $\frac{AD + DB}{DB} = \frac{AE + EC}{EC}$,

or, in the first figure, $\frac{AB}{DB} = \frac{AC}{EC}$.

This property, you will observe, has already been prov in the middle figure of Proposition 24.

Again, since $\frac{DB}{AD} = \frac{EC}{AE}$,

$$\therefore \quad \frac{DB}{AD} + 1 = \frac{EC}{AE} + 1.$$

That is, $\frac{DB + AD}{AD} = \frac{EC + AE}{AE}$,

or, in the first figure, $\frac{AB}{AD} = \frac{AC}{AE}$.

***Considering the first figure of Proposition 26, the i portant results*** $\frac{AB}{BD} = \frac{AC}{CE}$ ***and*** $\frac{AB}{AD} = \frac{AC}{AE}$ ***are regard as corollaries of the proposition.***

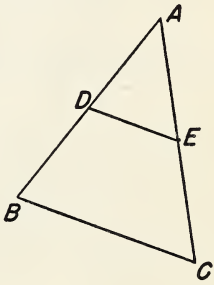
## EXERCISES

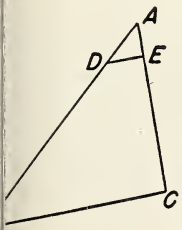
1. $P$ is a point in $AB$ a side of $\triangle ABC$. $Q$ is a point in such that $PQ \parallel BC$. If $AP = 1$ unit, $PB = 3$ units, and $= 2$ units, find the length of $QC$.

2. $M$ and $N$ are points in the sides $AB$ and $AC$ respectively $\triangle ABC$ such that $MN \parallel BC$. $AM = 4$ cm., $AB = 7$ cm., $= 1\frac{1}{2}$ cm. Find the length of $AC$.

3. $K$ and $L$ are points in the sides $AB$ and $AC$ respectively $\triangle ABC$ such that $KL \parallel BC$. $AK = 4$ units, $AC = 5$ ts, and $AL = 2$ units. Calculate the length of $AB$.

4. *In* $\triangle ABC$, $AD = DB$ *and* $DE$ $BC$. *Prove that* $E$ *is the middle point* $AC$.

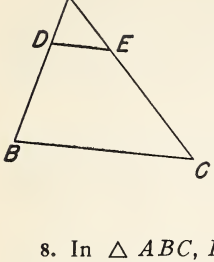



5. (*a*) In $\triangle ABC$, $AD$ is one-fifth of $AB$ and $DE \parallel BC$. Prove that $AE$ is one-fifth of $AC$.

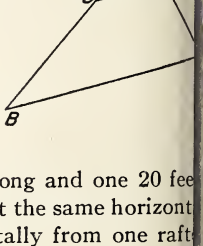
(*b*) If $AE = 4$ units, calculate the length of $EC$.

6. Devise a method based on Exercise 5 of dividing a given into any given number of equal parts.




7. In $\triangle ABC$, $DE \parallel BC$,
$DA = x + 1$ units,
$DB = x + 2$ units,
$AE = x + 3$ units,
and $EC = x + 5$ units.
Find $x$ and hence the length of $AC$.

8. In $\triangle ABC$, $DE \parallel BC$,
$AD = x$ units,
$DB = 3y$ units,
$AE = x + 1$ units,
$EC = 4y$ units,
$AB = 9$ units.
Calculate the length of $AC$.



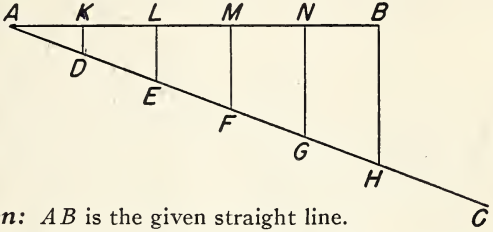

9. In a barn two rafters, one 30 feet long and one 20 fee are joined at the top and rest on two sills at the same horizont level. A wooden brace is nailed horizontally from one raft to the other. If it is nailed 6 feet from the lower end of t 30-foot rafter, how far from the lower end of the other raft is it nailed?

10. A telegraph pole 45 feet high broke at a height of 20 fe and the upper part bent over to the ground so that the t parts formed two sides of a triangle. The pole had stirru for climbing, one of which was 4 feet from the bottom a another of which was 5 feet from the top. If now a rope stretched between these two stirrups, what is the direction the rope?

11. (*a*) ***State the converse of Proposition 26.*** (*b*) ***Gi an indirect proof of this converse.***

### PROPOSITION 27. PROBLEM

***To divide a given straight line into any required mber of equal parts.***




***Given:*** $AB$ is the given straight line.

***Required:*** To divide $AB$ into a certain number of equal ts. Let us say that the required number of equal parts is :.

***Construction:*** At $A$ draw $AC$ making any angle with $AB$. $AC$ take any point $D$. Cut off from $DC$, $DE$, $EF$, $FG$, and , each of which is equal to $AD$.

Join $HB$. Through $G$, $F$, $E$ and $D$, draw $GN$, $FM$, $EL$ l $DK$ parallel to $HB$.

Then $AK = KL = LM = MN = NB$.

***Proof:*** In $\triangle AEL$, $DK \parallel EL$.

$$\therefore \frac{AD}{DE} = \frac{AK}{KL}.$$

But $AD = DE$, (Why?)

$$\therefore AK = KL.$$

In $\triangle AFM$, $EL \parallel FM$.

$$\therefore \frac{AE}{EF} = \frac{AL}{LM}.$$

But $AE = 2EF$, (Why?)

$$\therefore AL = 2LM.$$

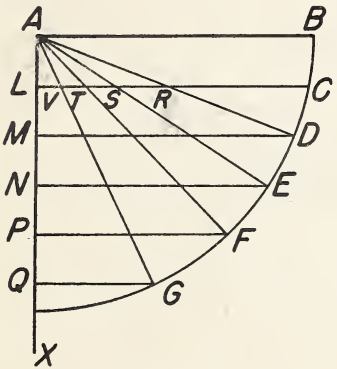
$$\therefore LM = AK = KL.$$

Similarly, it may be proved that $AK = KL = LM = MN$ $NB$.

## EXERCISES

1. Give an alternative construction and proof for Propos tion 8.

2. How would you obtain a line 3/7 inch in length?




3. $AB$ is a given line an $AX \perp AB$. From $AX$ equal parts $AL$, $LM$, $MN$ $NP$ and $PQ$ are cut off. Wit $A$ as centre and radius $AP$ a quadrant of a circle drawn.

$LC$, $MD$, $NE$, $PF$ and $QG$ are drawn $\| AB$.

$AD$, $AE$, $AF$, $AG$ are joine

(*a*) Show that $AR = \frac{1}{2}AB$, $AS = \frac{1}{3}AB$, $AT = \frac{1}{4}AB$ and $AV = \frac{1}{5}AB$.

(*b*) Find a line which is $\frac{4}{5}\ AB$.

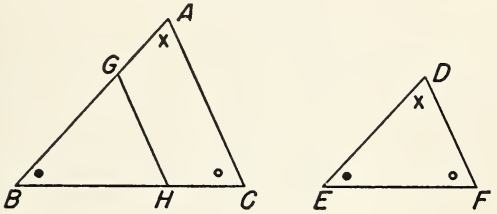
(*c*) Show that $AT - AV = \frac{1}{20}\ AB$.

(*d*) How would you obtain from the given figure a li equal to one-sixth of $AB$?

4. If there are three or more parallel straight lines and th intercepts made by them on one transversal are equal, th the corresponding intercepts on any other transversal are al equal.

### PROPOSITION 28. THEOREM

***If the angles of one triangle are respectively equal to angles of another, the corresponding sides of the angles are proportional.***




***Given:*** In $\triangle$s $ABC$, $DEF$, $\angle A = \angle D$, $\angle B = \angle E$, and $\angle C = \angle F$.

***Required:*** To prove that $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$.

***Proof:*** Apply $\triangle DEF$ to $\triangle ABC$ so that $\angle E$ coincides with $\angle B$ and $ED$ falls along $BA$.

Let $GBH$ be the new position of $\triangle DEF$.

$\angle BGH = \angle A$.

$\therefore GH \parallel AC$. (Why?)

$\therefore \frac{AB}{BG} = \frac{CB}{BH}$. (Why?)

$\therefore \frac{AB}{DE} = \frac{BC}{EF}$.

Since $AB$ and $DE$, $BC$ and $EF$ are any pairs of corresponding sides in the $\triangle$s $ABC$ and $DEF$ respectively, it follows that

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}.$$

* * *

COROLLARY: ***If the angles of one triangle are respectively equal to the angles of another, the sides that include equal angles are proportional.***

$$\frac{AB}{DE} = \frac{BC}{EF},$$

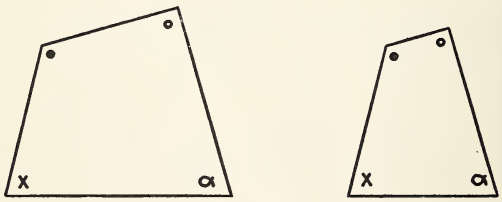
$$\therefore \frac{AB}{BC} = \frac{DE}{EF}.$$

Similarly, $$\frac{BC}{CA} = \frac{EF}{FD}.$$

and $$\frac{CA}{AB} = \frac{FD}{DE}.$$

**When are figures similar?** If two triangles have equal angles, they have the same shape. Provided that they are not congruent, the one is an enlargement of the other.

However, if polygons with more than three sides have equal angles, it does not necessarily follow that they are of the same shape. The two quadrilaterals in the figure illustrate this fact.




DEFINITION: ***Two polygons are similar when (1) the angles of one taken in order around the figure are respectively equal to the angles of the other taken in order; (2) the corresponding sides are proportional.***

Why is it sufficient to state that the angles are respectively equal when defining similar *triangles*?

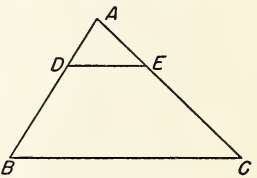
It is unnecessary to memorize the definition of similar ures. You should know, however, that two triangles are ilar if their angles are equal, but that this fact is not enough establish the similarity of other polygons.

### EXERCISES

1. State whether all examples of each of the following ures are similar: (*a*) equilateral triangle, (*b*) parallelogram, rectangle, (*d*) square, (*e*) rhombus, (*f*) trapezium.

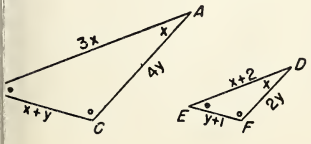
2. If two triangles have two angles of one respectively al to two angles of the other, prove that they are similar.

3. In $\triangle ABC$, $D$ is a point on $AB$ h that $AD = \frac{1}{3}AB$, and $E$ is on $AC$ h that $DE \parallel BC$. (*a*) Show that $= \frac{2}{3}AC$. (*b*) If $AE = 5$ units, find length of $EC$.




4. In a $\triangle ABC$, $AB = 5$ cm., $= 8$ cm. A line $DE$ parallel to $BC$ s $AB$ at a point $D$, 2 cm. from $A$, and cuts $AC$ at $E$. d (*a*) the ratio $\frac{AE}{EC}$, (*b*) the length of $AE$.

5. $X$ and $Y$ are points in the sides $PQ$ and $PR$ respec- ly of $\triangle PQR$ such that $XY \parallel QR$. $PY = 3$ units, $PR$ 7 units, and $XQ = 12$ units. Find (*a*) the ratio $\frac{PX}{PQ}$, (*b*) length of $PX$.




6. From the data in the figures, find the number of units of length in (*a*) $BC$, (*b*) $EF$.

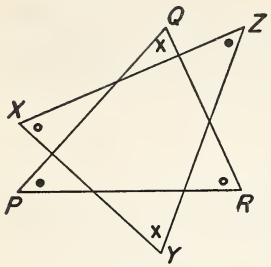



Fig. A

7. In your work-book, wri
the ratio which will complete ea
of the following incomplete stat
ments of proportion:

(*a*) $\frac{ZX}{XY} = \text{———}$;

(*b*) $\frac{XZ}{ZY} = \text{———}$;

(*c*) $\frac{XY}{YZ} = \text{———}$. (Fig. A

8. In $\triangle ABC$, $\angle BAC = 90°$ and $AD \perp BC$. (*a*) Prove that the three triangles $ABC$, $ABD$, and $ADC$ are similar. (*b*) Prove that $\frac{AB}{BC} = \frac{DB}{BA}$, and hence that the square on $AB$ is equal to the rectangle $BD.BC$. (*c*) Prove that $CA^2 = CD.CB$. (*d*) Prove that $AD^2 = BD.DC$ (By $CA^2$ is meant the number of units of area in a square whose side $CA$. See p. 179.)

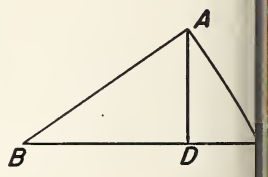
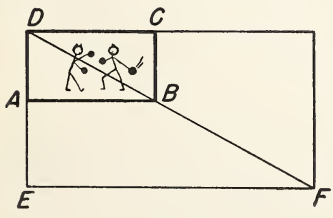



Fig. B




9. A photographer wis
ed to enlarge the pictu
$ABCD$ so that the ne
width would be $D$
Through $E$, he drew a li
$\| AB$ to meet the diagor
$DB$ produced at $F$. He to
$EF$ as the length of the e
largement. Show that t
new length is to the ne
width as the old length is
the old width.

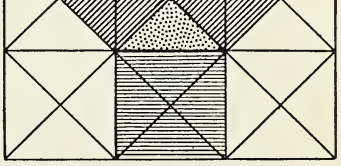
10. If the length and width of a rectangular photograph 5 inches and $3\frac{1}{2}$ inches respectively, what is the new width he photograph is enlarged so that its length is 12 inches?

11. A young man wished to know if a portion of his dis-rge certificate from the navy could be "photographed vn" so that it would fit in his bill-fold. The dimensions of part in question were $7\frac{1}{4}$ in. $\times$ $6\frac{1}{8}$ in. The dimensions of bill-fold were $3\frac{5}{8}$ in. $\times$ $2\frac{5}{8}$ in. Could the part be reduced as to fit in the bill-fold exactly?

## *The Pythagorean Theorem*

No doubt, many of you know that there has been discussion the possibility that life exists on Mars. A proposal was de that a signal be sent to show the Martians, if any, that igs of some intelligence dwell on the planet Earth. There a few minor difficulties. What kind of message would ry meaning to the inhabitants of another planet? Someone ught of signalling a certain geometrical figure, that for the orem named after Pythagoras. The signal might be a le lacking in warmth, but it would make a first-rate letter-d for intelligence corresponding with intelligence.

Let us tell you something about the famous theorem of hagoras.



**Learning from tiles.** enty-five or more cen-ies ago, a man sat staring a tiled floor. He noticed t the largest of the three ded squares had the ne area as the two small-ones combined. That sn't very exciting; the lest slave in the tilery knew all about it. However, he

was not discouraged; he kept on noticing in a persiste manner. Next, he observed that the large square was t square on the longest side of a right triangle and that t smaller squares were the squares on the other two sides. very remarkable thing!

But is this true of all right triangles or is it true only of t special case where the right triangle is isosceles? Read o next installment.

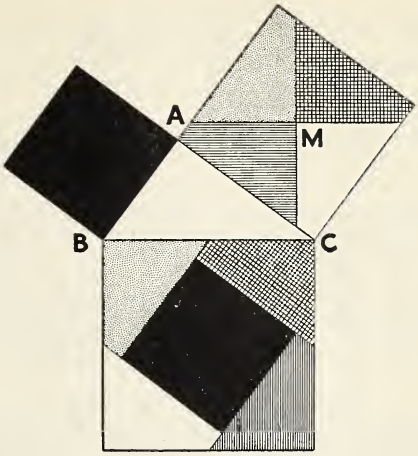
**An experiment with right triangles.** It is suggested th the members of the class construct right triangles and c operate in completing in the work-books a table such as t following one. For the sake of convenience, in each triang the angle *C* should be the right angle. The sides may be any length.

| *Angle C* | *Length of side* | | | *Area of square on AC* | *Area of square on BC* | *Sum of areas of squares on AC and BC* | *Area o square on AE* |
|---|---|---|---|---|---|---|---|
| | *AC* | *BC* | *AB* | | | | |
| 90° | | | | | | | |
| 90° | | | | | | | |
| 90° | | | | | | | |

What inference may be made from the table? Do n generalize rashly.

* * *

You may be interested in the following paper-cutti demonstration. Thin cardboard is preferable to paper.



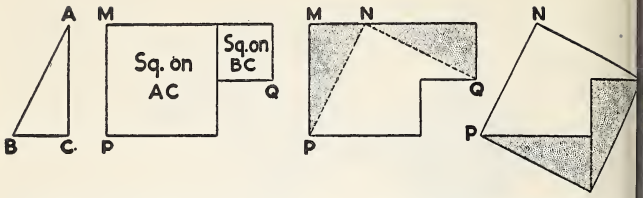

Using any instruments you please, construct any right ngle $ABC$. On each side make a square.

Find the point $M$ where the diagonals of the square on intersect. Through $M$, draw a line parallel to $BC$ and a perpendicular to $BC$.

Cut out the square on $AB$ and also the four pieces into ch the square on $AC$ is divided. If these be assembled to n a square, to what square is it equal?

There is evidence that this demonstration was used by the nese more than a thousand years B.C.

Neither your experiment nor your paper-cutting can serve roof of the statement that *the square on the longest side of a t triangle is equal to the sum of the squares on the other two s.* (The longest side in a right triangle is called the **otenuse.**)

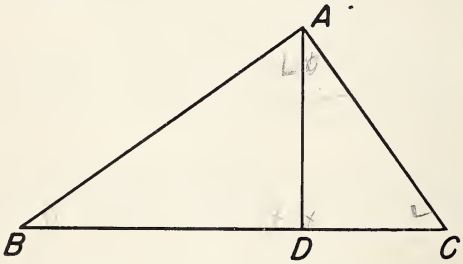



Another experimental demonstration. The third figure indicates how area equal to the sum of the squares on $AC$ and $BC$ can be cut to make fourth figure, which is equal to the square on $AB$.

We shall now give a proof of the proposition. More th three hundred proofs are known today. When the first pr was discovered, oxen were sacrificed to the gods.

### PROPOSITION 29. THEOREM

***If a triangle contains a right angle, the square on t hypotenuse is equal to the sum of the squares on t other two sides.***




***Given:*** $ABC$ is a triangle in which $\angle BAC = 90°$.

***Required:*** To prove that $BC^2 = AB^2 + AC^2$.

***Proof:*** From $A$, draw $AD \perp BC$.

In $\triangle$s $ABC$, $ABD$,

$\angle BAC = \angle BDA$, (Why?)

$\angle B$ is common,

$\angle C = \angle BAD$, (Why?)

$\therefore$ $\triangle$s $ABC$ and $ABD$ are similar.

$$\therefore \frac{AB}{BC} = \frac{DB}{BA}. \qquad \text{(Why?)}$$

$$\therefore AB^2 = BD.BC.$$

Similarly, it may be shown that $\triangle$s $ABC$ and $ADC$ are lar.

$$\therefore \frac{BC}{CA} = \frac{AC}{CD}.$$

$$\therefore AC^2 = CD.BC.$$

But $$AB^2 = BD.BC.$$

$$\therefore AB^2 + AC^2 = BD.BC + CD.BC,$$
$$= BC\ (BD + CD),$$
$$= BC.BC,$$
$$= BC^2.$$

*mple* 1. Given that the two short sides of a right triangle 3 inches and 4 inches, find the length of the hypotenuse.

Let the number of inches in the length of the hypotenuse be $x$.

Then $x^2 = 3^2 + 4^2$,
$= 9 + 16$,
$= 25$.

Therefore, $x = 5$.

Hence, the length of the hypotenuse is 5 inches.

*mple* 2. Let us calculate the length of the side $AC$ he $\triangle ABC$.

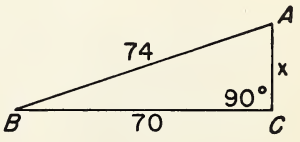
Let $AC$ contain $x$ units of length.

$AB^2 = BC^2 + CA^2$.

That is, $74^2 = 70^2 + x^2$.

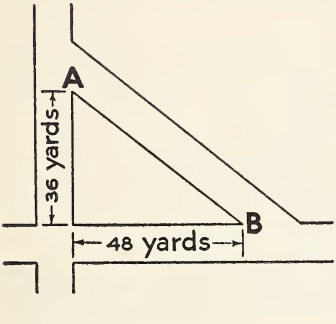
Hence, $x^2 = 74^2 - 70^2$
$= (74 + 70)\ (74 - 70)$
$= 144 \times 4$.

$\therefore x = 24$.




Hence, $AC$ contains 24 units of length.

## EXERCISES




1. (*a*) How many ya
long is the path $AB$?
(*b*) What distance is s
ed in using the short-cut?

2. A 25-foot ladder
leaning against a wall.
the foot of the ladder i
feet from the wall, at w
height above the ground
the top of ladder?

3. Can a knitting nee
fourteen inches long be l
on the bottom of a rectan
lar box one foot long and
inches wide? Explain.

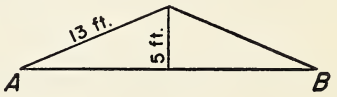
4. $ABC$ is a triangle in which $\angle B = 90°$. Copy
following table and complete the entries. In doing this,
may use the table of squares and square roots given on page 2

| *Side* | *Number of units of length* | | | | | |
|---|---|---|---|---|---|---|
| $AB$ | 5 | 15 | | | 60 | 12 |
| $BC$ | 12 | | 40 | 7 | | 35 |
| $CA$ | | 17 | 41 | 25 | 61 | |

5. $ABC$ is a triangle in which $\angle A = 90°$. The numbers nits of length in two sides are given. Calculate the number nits in the third side.

| | $AB$ | $BC$ | $CA$ |
|---|---|---|---|
| $(a)$ | 6 | . . . | 8 |
| $(b)$ | 5 | 13 | . . . |
| $(c)$ | . . . | 25 | 24 |
| $(d)$ | 9 | 15 | . . . |
| $(e)$ | . . . | $m$ | $n$ |
| $(f)$ | $c - 1$ | $c + 1$ | . . . |
| $(g)$ | $2d$ | . . . | $d - 1$ |
| $(h)$ | $m - n$ | . . . | $2\sqrt{mn}$ |
| $(i)$ | $2pq$ | $p^2 + q^2$ | . . . |
| $(j)$ | $2n$ | . . . | $n^2 - 1$ |

6. Find the width $AB$ of roof shown in the diagram.



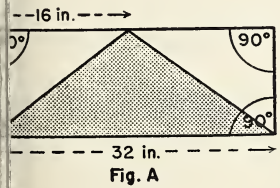



Fig. A

7. Calculate $(a)$ the perimeter, $(b)$ the area of the shaded portion of Figure $A$.

8. Calculate the area of shaded portion of Figure

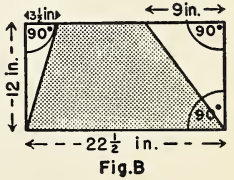
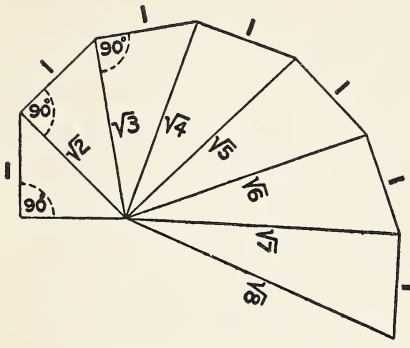


Fig.B

9. Draw a line $\sqrt{61}$ centimetres long. (*Hint*: $36 + 25 =$




10. (*a*) Exam
the accompany
figure and explain
construction.

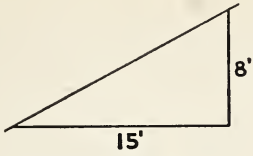
(*b*) How wo
you use this constr
tion to find lines
length $\sqrt{10}$ inc
and $\sqrt{11}$ inches?

11. Winnifr
Wheeler and P
cilla Pedlar rode
miles southeast,
their lunch and then rode 6 miles due west. They were t
due south of the point from which they originally set
How far were they from that point?

12. (*a*) Find the ratio of the length of the diagonal
square to the length of a side.

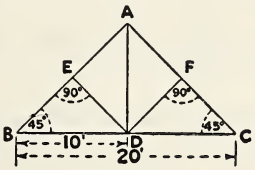
(*b*) Does this ratio hold good for all squares? Why?

(*c*) Show that the square on the diagonal of a given sq
has twice the area of the given square.




13. Calculate the length of the rafter, allowing 9 inches for ection at each end.

14. Find the lengths of $DE$, $DF$, $AD$, $AE$, and $AF$.




15. Use the table on page 221 to find (*a*) $29^2$, (*b*) $26^2$, (*c*) (*d*) $\sqrt{1089}$, (*e*) $\sqrt{1681}$, (*f*) $\sqrt{961}$.

16. Prove that the perpendicular to a line from a point ide the line is the shortest line that can be drawn from the t to the line. (*The length of the perpendicular is called the ince from the point to the line.*)

17. Construct a square equal to the sum of three given ires.

18. Construct a square equal to the difference of two given ires.

19. If the sides of a rectangle are in the ratio 1 to 2, show the square on a diagonal is 5 times the square on the ter side.

20. $ABC$ is a triangle in which $AB = BC$ and $\angle B = 90°$. Prove that $AC^2 = 2AB^2$. (*b*) Find the ratio of $AB$ to $AC$.

21. Produce a given line $AB$ to $C$, so that $BC^2 = 2AB^2$.

**22.** $ABC$ is an equilateral triangle in which the perp dicular from $A$ to $BC$ meets $BC$ at $M$. $BM$ contains $p$ u of length. (*a*) Find $AM$ in terms of $p$. (*b*) Find the ratio $BM$ to $AB$. (*c*) Find the ratio of $AM$ to $AB$.

**23.** $ABC$ is a triangle in which $\angle C = 90°$. $D$ is a point on $AC$. Show that $AC^2 + BD^2 = AB^2 + CD^2$.

**24.** A straight line $MN$ is produced to $P$ so that $MP$ $3MN$. An equilateral triangle $QNP$ is drawn on $NP$. Sh that $MQ^2 = 7MN^2$.

**25.** $P$ is any point in a rectangle $ABCD$. Show t $AP^2 + CP^2 = BP^2 + DP^2$.

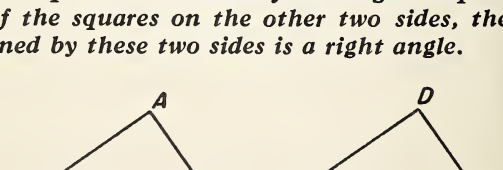
**26.** *If the hypotenuse and one side of a right triangle equal to the hypotenuse and one side of another right triangle, triangles are congruent.* (This conclusion may be used with proof in succeeding exercises.)

**27.** Prove that the bisectors of the angles of a triangle concurrent.

**28.** State the converse of the Pythagorean theorem.

## PROPOSITION 30. THEOREM

***If the square on one side of a triangle is equal to sum of the squares on the other two sides, the an contained by these two sides is a right angle.***



***Given:*** $ABC$ is a $\triangle$ in which $BC^2 = AB^2 + AC^2$.

***Required:*** To prove that $\angle A = 90°$.

***Proof:*** Construct a right angle $D$ and cut off $DE$ equal to $AB$, and $DF$ equal to $AC$. Join $EF$.

$$\begin{aligned} BC^2 &= AB^2 + AC^2 && \text{(Why?)} \\ &= DE^2 + DF^2 && \text{(Why?)} \\ &= EF^2. && \text{(Why?)} \\ \therefore BC &= EF. \end{aligned}$$

In $\triangle$s $ABC$, $DEF$ $\begin{cases} AB = DE, & \text{(Why?)} \\ AC = DF, & \text{(Why?)} \\ BC = EF. & \text{(Why?)} \end{cases}$

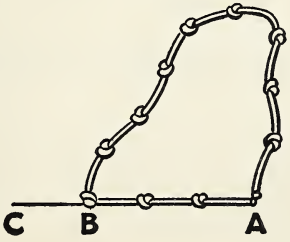
$$\therefore \angle A = \angle D. \quad \text{(Why?)}$$

But $\angle D = 90°$,

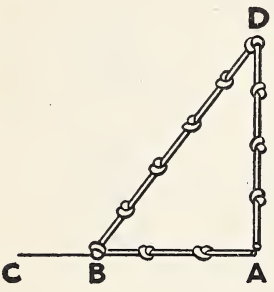
$$\therefore \angle A = 90°.$$

*The Pythagorean theorem and its converse are two of the most important facts which you will learn in geometry. They are described in an old book as "of infinite use in most Geometrical Conclusions".*

**The rope stretchers.** When the Egyptians wished to construct a right triangle out of doors, they used a loop of



rope with eleven equally spaced knots. If they required perpendicular to the line $CA$ at the point $A$, they fasten both ends of the rope at $A$ and stretched part of the r along $CA$. At $B$, three spaces from $A$, the rope was a fastened. Then, some held the rope at $D$, spaces from $B$ and pulled rope taut. $DAB$ was a angle with sides of 3, 4 a units. The angle $A$ oppo to the longest side wa right angle. (Converse the Pythagorean theore This method of construc a right angle out of doo still sometimes used.




## EXERCISES

1. In $\triangle ABC$, $AB = 0.7$ cm., $BC = 2.4$ cm., $CA =$ cm. Prove that $\angle B = 90°$.

2. In $\triangle ABC$, $AB = 8.5$ in., $BC = 1.3$ in., $CA = 8.$ Prove that $\angle C = 90°$.

3. In $\triangle ABC$, $AB = m^2 - n^2$ units, $BC = m^2$ units, $CA = 2mn$ units. Prove that $\angle A = 90°$.

4. A loop of rope 80 feet long is stretched into a trian shape with sides 30 ft., 16 ft. and 34 ft. What kind of tri is it?

5. Of three isosceles right triangles one is equal in ar the sum of the other two. Show that, if one of the e sides be taken from each triangle, the three lines ca assembled into a right triangle.

## TABLE OF SQUARES AND SQUARE ROOTS

| | $n^2$ | $\sqrt{n}$ | $n$ | $n^2$ | $\sqrt{n}$ | $n$ | $n^2$ | $\sqrt{n}$ | $n$ | $n^2$ | $\sqrt{n}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1.000 | 16 | 256 | 4.000 | 31 | 961 | 5.568 | 46 | 2116 | 6.782 |
| 2 | 4 | 1.414 | 17 | 289 | 4.123 | 32 | 1024 | 5.657 | 47 | 2209 | 6.856 |
| 3 | 9 | 1.732 | 18 | 324 | 4.243 | 33 | 1089 | 5.745 | 48 | 2304 | 6.928 |
| 4 | 16 | 2.000 | 19 | 361 | 4.359 | 34 | 1156 | 5.831 | 49 | 2401 | 7.000 |
| 5 | 25 | 2.236 | 20 | 400 | 4.472 | 35 | 1225 | 5.916 | 50 | 2500 | 7.071 |
| 6 | 36 | 2.449 | 21 | 441 | 4.583 | 36 | 1296 | 6.000 | 51 | 2601 | 7.141 |
| 7 | 49 | 2.646 | 22 | 484 | 4.690 | 37 | 1369 | 6.083 | 52 | 2704 | 7.211 |
| 8 | 64 | 2.828 | 23 | 529 | 4.796 | 38 | 1444 | 6.164 | 53 | 2809 | 7.280 |
| 9 | 81 | 3.000 | 24 | 576 | 4.899 | 39 | 1521 | 6.245 | 54 | 2916 | 7.348 |
| 0 | 100 | 3.162 | 25 | 625 | 5.000 | 40 | 1600 | 6.325 | 55 | 3025 | 7.416 |
| . | 121 | 3.317 | 26 | 676 | 5.099 | 41 | 1681 | 6.403 | 56 | 3136 | 7.483 |
| 2 | 144 | 3.464 | 27 | 729 | 5.196 | 42 | 1764 | 6.481 | 57 | 3249 | 7.550 |
| 6 | 169 | 3.606 | 28 | 784 | 5.292 | 43 | 1849 | 6.557 | 58 | 3364 | 7.616 |
| 4 | 196 | 3.742 | 29 | 841 | 5.385 | 44 | 1936 | 6.633 | 59 | 3481 | 7.681 |
| 5 | 225 | 3.873 | 30 | 900 | 5.477 | 45 | 2025 | 6.708 | 60 | 3600 | 7.746 |

## CHAPTER XI

# REVIEW SUMMARIES

### *I. When are two triangles congruent?*

Two triangles are congruent:

1. when two sides and the contained angle of one a respectively equal to two sides and the contained angle of t other; (S.A.S.)[1] (Prop. 1, page 95)

2. when the three sides of one are respectively equal the three sides of the other; (S.S.S.)[1] (Prop. 3, page 110)

3. when two angles and a side of one are respectiv equal to two angles and a side of the other. (A.A.S. or A.S.A (Prop. 20, page 172)

4. when they are right triangles with the hypotenuse a a side of one equal to the hypotenuse and a side of the oth (Ex. 26, page 218)

### *II. When are two triangles equal in area?*

Two triangles are equal in area

1. when they are congruent;

2. when they are on the same base and between the sa parallels; (Prop. 23, page 184)

3. when they stand on equal bases and have equal a tudes. (Corollary of Prop. 24, page 185)

### *III. When are two straight lines equal?*

Two straight lines are equal:

1. when they are corresponding sides in congruent angles;

[1]The letters S.A.S. are sometimes used as a reminder of the congru of two triangles which have two sides and the contained angle of one equ the corresponding parts of the other (Prop. 1, 1 above). Similarly, Pro 2 above is symbolized by S.S.S. and Prop. 1, 3 by A.A.S. or A.S.A.

2. when they result from the bisection of a line; (Prop. 8, ;e 128)

3. when they are two of any number of equal parts into ch a line is divided; (Prop. 27, page 203)

4. when they are opposite sides of a parallelogram; (Ex. 8, e 174)

5. when they are sides opposite equal angles in a triangle. p. 21, page 177)

***When are two angles equal?***

Two angles are equal:

1. when they are the complements of equal angles; inition)

2. when they are the supplements of equal angles; finition)

3. when they are corresponding angles in congruent ngles;

4. when they are opposite equal sides in a triangle; )p. 2, page 106)

5. when they are formed by bisecting a given angle; (Prop. )age 122)

6. when they are vertically opposite; (Prop. 10, page 137)

7. when they are alternate angles formed with a trans- al by two parallel lines; (Prop. 16, page 152)

8. when they are corresponding angles formed when a sversal cuts two parallel lines; (Prop. 17, page 154)

9. when they are the opposite angles in a parallelogram; ercise 8, page 156)

10. when they are the third pair of angles in two triangles hich two angles of the one are respectively equal to two les of the other. (Ex. 1, page 162)

### *V. When are two straight lines parallel?*

(The making of this summary is left as an exercise for t student.)

### *VI. What are some of the properties of parallel straig lines?*

1. The alternate angles made with a transversal are equ (Prop. 16, page 152)

2. The corresponding angles made with a transversal equal. (Prop. 17, page 154)

3. The interior angles on the same side of the transver are supplementary. (Prop. 18, page 154)

4. Straight lines which are perpendicular to the sa straight line are parallel to one another. (Ex. 4, page 155

5. Straight lines which are parallel to the same strai line are parallel to one another. (Ex. 7, page 156)

6. If there are three or more parallel straight lines and intercepts made by them on one transversal are equal, t the corresponding intercepts on any other transversal are equal. (Ex. 4, page 204)

7. A straight line drawn through the midpoint of one s of a triangle parallel to the base bisects the third side. ( 6, page 201)

8. The straight line joining the midpoints of two sides triangle is parallel to the third side and equal to half o (Ex. 12, page 175)

### *VII. What are some of the properties of a parallelogr*

1. The opposite angles of a parallelogram are eq (Ex. 8, page 156)

2. The opposite sides of a parallelogram are equal. 9, page 174)

3. If both pairs of opposite angles of a quadrilateral are al, the figure is a parallelogram. (Ex. 4, page 167)

4. The diagonals of a parallelogram bisect each other. . 10, page 174)

5. If both pairs of opposite sides of a quadrilateral are al, the figure is a parallelogram. (Ex. 11, page 157)

6. If the diagonals of a quadrilateral bisect each other, the re is a parallelogram. (Ex. 9, page 156)

7. Each diagonal divides the parallelogram into congruent ngles. (Ex. 8, page 174)

8. Parallelograms on the same base and between the same allels are equal in area. (Prop. 22, page 180). (This position yields the corollary that a rectangle and a parallelon on the same base and between the same parallels are al in area.) (Page 181)

* * *

It is suggested that you make a list of propositions and the e important exercises dealing with topics such as the owing:—

***Properties of special triangles***

1. The triangle with two equal sides.
2. The equilateral triangle.
3. The right triangle.
4. The isosceles right triangle.

***Properties of special parallelograms***

1. The rhombus.
2. The rectangle.
3. The square.

### *VIII. Constructions with which you should be familia*

1. At a given point in a given straight line, to constru an angle equal to a given angle. (Prop. 4, page 121)

2. To bisect a given angle. (Prop. 5, page 122)

3. To draw a perpendicular to a given straight line a given point in the line. (Prop. 6, page 125)

4. To draw a perpendicular to a given straight line fro point outside the line. (Prop. 7, page 126)

5. To draw the right bisector of a given straight li (Prop. 8, page 128)

6. To construct a triangle when given the three si (Prop. 9, page 130)

7. To construct a triangle when given two sides and included angle. (Page 93)

8. To construct a triangle when given one side and t angles. (Ex. 1, page 173)

9. To construct a triangle when given two sides and angle not included. (Ex. 5, page 104)

10. To construct with straight edge and compasses following angles:—(i) $7\frac{1}{2}°$, (ii) $22\frac{1}{2}°$, (iii) 30°, (iv) 45°, (v) 6 (vi) $67\frac{1}{2}°$, (vii) 75°, (viii) 90°, (ix) $112\frac{1}{2}°$, (x) 120°, (xi) 135°

11. Through a given point, to draw a straight line para to a given straight line. (Prop. 15, page 150)

12. To construct a square on a given straight line. ( 16, page 175)

13. To construct a rectangle given the length and wi (Ex. 13, page 157)

14. To construct a parallelogram given two adjacent si and an angle. (Ex. 14, page 158)

5. To construct a quadrilateral given the four sides and angle. (Ex. 4, page 131)

6. To divide a given straight line into any required ber of equal parts. (Prop. 27, page 203)

CHAPTER XII

# YOUR GEOMETRY IN RETROSPECT

*"If a man's wit be wandering, let him study the mathematics; for in demonstrations, if his wit be called away never so little, he must begin again."* Bacon

## *Do we know what we are talking about geometry?*

The great importance of definitions, not only in geomet but in the discussions of everyday life has been pointed o However, if we try to define every term, we run into troub For instance, if we take as the definition of 'equilateral triang *a triangle in which all the sides are equal*, we have used terms 'triangle', 'sides', and 'equal'. All these require defi tion. In defining 'triangle', we must use the phrase 'straig line'. The definition of 'straight line' involves the te 'line'. If we define 'line' as *length without breadth*, what 'length' and 'breadth'? If we define 'length' as 'distan what is 'distance'? Somewhere this must come to an e The end point is an undefined term. Had we not stopp we should have found it necessary to bring into our definit some term defined at an earlier stage. You will recall th when this happens, we have a circular definition—which is n definition. It is, of course, reasonable to expect that meaning of an undefined term is clearer than the meaning any term into the definition of which it enters.

> *In geometry, indeed in all mathematics, the definition of each term rests on one or more undefined terms.*

That is why Bertrand Russell referred to mathematics as e subject in which we never know what we are talking ut".

## *Assumptions again!*

We have seen that if the process of definition is pushed her and farther back, we arrive eventually at terms which st be accepted without being defined.

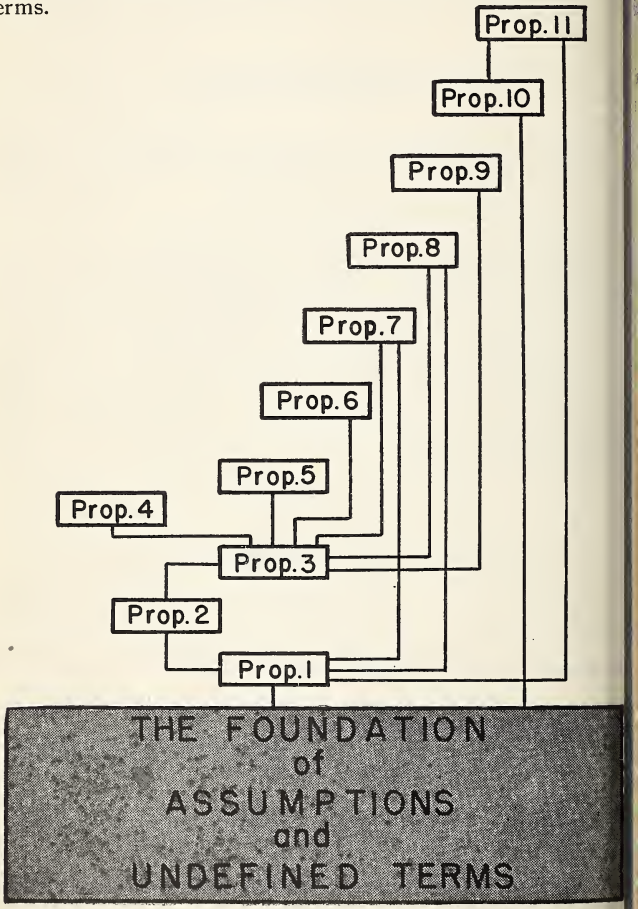
A parallel situation is found in mathematical propositions. prove Proposition 3, we had to call to our aid the pre-usly proved Propositions 2 and 1 as well as a certain assump-n. To prove Proposition 2, it was necessary to make use he conclusion of Proposition 1 and to introduce an assump-n (namely, that an angle can be bisected). In our proof Proposition 1, we had, of course, no previously proved positions on which to depend, but we did have to make an umption (page 102).

*These assumptions are unproved propositions.*

Thus, *the foundations of mathematics are unproved statements ch involve certain undefined terms.*

To sum up: In mathematics, we begin with undefined ns and assumptions. Other terms are defined with the of the undefined terms. By correct (or valid) reasoning, clusions (propositions) are derived from the assumptions. m these conclusions and the assumptions, new conclusions obtained. In turn, these conclusions and assumptions d other conclusions. And so the chain of mathematical soning develops.

The following diagram indicates the relationship of th first eleven propositions and the assumptions and undefine terms.



### EXERCISE

Continue the diagram of p. 230 until it embraces the twenty propositions.

*Since the whole edifice of mathematics is erected on a founda- of assumptions, the truth of the vast number of conclusions onditional upon the truth of these assumptions.*

> *This dependence upon assumptions exists in every reasoning process. Need we wonder that careful thinkers admit no more assumptions than are essential and that they challenge the truth of every one admitted?*

How are the tremendously important assumptions of netry justified? Much has been written on this subject; e has been controversy. It has been stated that each mption is self-evident, "assented to as soon as mentioned", easy and evident that no person will hesitate to allow it" yfair). It has been asserted that each is a generalization n experience. In other words, we learn that "if two ght lines intersect, they do so in one point only" in the we learn that corns are caused by tight shoes. The ern view is that the assumptions are hypotheses (intelli- guesses) concerning the nature of space.

For centuries, the assumptions of Euclid were regarded as ssailable truths. In modern times, a more critical view been taken by mathematicians: The "axiom of parallels". ch has been expressed in different ways,[1] was regarded as icularly unsatisfactory. Lobatchewski discarded this as-

[1] For example: *Through a given external point, one, and only one, line can wn parallel to a given line,* and *Two intersecting lines cannot both be el to a third line.*

sumption and, substituting another for it, built up a logica geometry very different from the old familiar one. Othe non-Euclidean geometries also have been derived by mathe maticians who made changes in the assumptions. We refe to this matter to emphasize the all-important bearing o assumptions on conclusions.

CHAPTER XIII

# SOME ERRORS TO BE AVOIDED IN REASONING

An acquaintance with some of the more common errors reasoning, or 'fallacies' as they are called, is useful to anyone wishes to be critical of his own thinking and of the arguments of those who seek to convince him.

## *Pitfall Number One*

From the reasonably well-grounded assumption *All ducks are swimmers*, no sensible person would infer that *All swimmers are ducks*. The second statement is the converse of the first. No matter how well a statement is established, the truth of its converse cannot be asserted without independent proof.

However, it is not always so easy to observe that the fallacy has been committed. Consider the following quotation from an imaginary speech:

"*It is a regrettable but an undeniable fact—and one to be kept clearly in mind—that every supporter of the Halcyon Party is a mere Elysian. This unfortunate state of affairs, long suspected by thoughtful and far-sighted public men, was dragged into the full light of day in the recent public investigation, when it was shown that without exception the Elysians were actively promoting the Halcyonist cause.*"

Our speaker accepts the results of the public investigation that *All Elysians are supporters of the Halcyon party* and unthinkingly concludes that *All supporters of the Halcyon party are Elysians*. It will be observed that he did not place these two

assertions side by side where comparison would be easy an detection of the fallacy inevitable.

This fallacy is the *assumption of a converse.*

From the rule, *If a person is a graduate, she is eligible vote*, it would be fallacious to conclude that *If a person eligible to vote, she is a graduate.* There might also be a ru that non-graduates with certain qualifications are eligible.

## *Pitfall Number Two*

Let us accept the following assertion:

*If Jean bought a new dress, then she went to the dance.* Th acceptance gives us no right to say that *If Jean did not buy a n dress, she did not go to the dance.* If we deny the "if" part, w are not justified in denying the "then" part.

In disguised form, this fallacy is contained in Ethel conversation, faithfully reported below.

Ethel (with partly concealed satisfaction): "*Patricia ca vote for the executive.*"

Mary: "*It is just like her not to pay the fee.*"

Ethel: "*It isn't a question of fees. All graduates are eligible vote, and Patricia, poor dear, is a non-graduate.*"

From the rule *All graduates are eligible to vote*, Ethel co cludes fallaciously that *All non-graduates are ineligible to vote.* would have been correct, but perhaps quite inapplicable Patricia, had she said, "*All ineligible to vote are non-graduates.*"

If we express the rule *All graduates are eligible to vote* in t form *If a person is a graduate, then she is eligible to vote*, it apparent that Ethel committed the fallacy of denying the "i clause. There are many Ethels.

## *Pitfall Number Three*

"Begging the question" is assuming the truth of a state-
ıt for which proof is required. The fallacy is a common
, often concealed in the thinking of people who are aware
ts reputation and who would be shocked at its presence.

In the following dialogue, we have an example of the fallacy
egging the question:

*'Without a doubt, A is a man of bad character."*

*'Why?"*

*'He is very intimate with B."*

*"What is the matter with B?"*

*"No friend of A could be straight."*

In the following conversation Mrs. Goscandle openly and
shamedly begs the question.

Mrs. Goscandle: *"In my opinion, he is quite as much to
ne as she."*

Miss Scandossip (helpfully): *"You really think so, Lyra?"*

Mrs. Goscandle: *"Any man who would marry the like of her
t worth much. Is he?"*

Miss Scandossip (savouring the morsel): *"Men are strange.
what is the matter with her?"*

Mrs. Goscandle (uplifting eyebrows and voice): *"Matter
her? Can't you imagine the type of woman who would
pt a proposal from that man?"*

There are a number of varieties of the question-begging
acy. Mrs. Goscandle's variety is the 'circular argument'.
e is another piece of reasoning that contains the fallacy.

Mrs. Tendersprout: *"Do you recommend this article?"*

Attendant: "*Look at the name, if you please, madam. Th name is on an article we know to be exceptionally good. Th inspires confidence in a trade name. Doesn't it, madam? We that confidence in the name gives both of us, customer and retail a feeling that we can't go far astray with the article that carri it.*"

Many questions provide examples of the question-beggi fallacy. In each of the following, the truth of something th has not been proved is taken for granted.

"*Why were you driving so quickly when the collision occurred*

"*Well, have you recovered from your tantrum yet?*"

"*Was it right to run away when you broke the window?*"

There are 'explanations' which do not explain because th beg the question. Such are the following:

*You can see through glass because it is transparent.*

*Opium induces sleep because it is a soporific.*

*Paris green is particularly useful for destroying codling-mo larvae because it is such a powerful insecticide.*

*Horses are spirited because they are so equine; cats graceful because there is something exquisitely feline about the dogs are faithful because of their essential caninity.*

It should not be overlooked that epithets often contain t fallacy of question-begging. If a lawyer refers to a man w is accused, but not proved guilty, of wife-beating as "t brutal scoundrel" he is committing the fallacy. When hear someone described as "an irresponsible visionary", policy as "intelligent and far-sighted", or an action as "despic ble", we should ask ourselves if the speaker has proved t applicability of his descriptions. If he has not done so,

, with perfect propriety, quietly attach to him the epithet question-beggar".

## *Pitfall Number Four*

A lawyer unable to produce conclusive evidence that a ndant had committed a shocking crime, devoted time and gy to impressing the judge with the hideous nature of the l.

In this example, there was no difficulty in detecting the fact t the lawyer was dealing with something that had no bearing is real task, namely to show that the defendant was guilty. description of the crime was not relevant to the proposition e proved. Frequently, the complete assurance of a speaker, shrouding of his reasoning in a fog of words, and the feeling arouses in his audience render it difficult to separate the vant from the irrelevant. Under these circumstances, the aker may indeed prove some point to the entire satisfaction himself and his listeners. Unfortunately, the point proved y not be the one for which proof is required.

The fallacy is one of the forms of 'befogging the issue'. ew additional illustrations will be given.

*A speaker set out to prove that secondary education was not eficial to the vast majority of those receiving it. He showed, is own satisfaction, that it was burdening the community with s.*

*An official, who had been detected in accepting a bribe, nded himself by showing that his action had not caused loss he government.*

*A speaker, defending a certain plan of government aid to the mployed, proved that help of some kind is necessary.*

There is a formula long used in calling attention to irrel vant conclusions—"That's beside the point".

## *Pitfall Number Five*

If a man fails to prove that he was not at the intersectio of Criss and Cross Streets at 8.30 on a certain evening, cannot be taken for granted that actually he was there. Failu to prove a contention does not disprove it, nor does it pro the opposite. Having undermined the case of his opponen an unscrupulous person may, through open assertion skilful suggestion, convey to the audience that this constitut proof of the correctness of his own case.

We may call this pitfall the fallacy of "Unproved, ther fore disproved".

## *Pitfall Number Six*

One reason why mathematics is included in the course study of secondary schools is to make apparent to young peop the great need, not only in mathematics, but in all discussion of precise meanings. How ridiculous it would be of businessm to discuss prices in pounds if it were not clearly understo whether the pounds were those of the United Kingdom, Australia, New Zealand or South Africa! To speak of to without agreement on whether the ton is a "long ton" of 22 pounds or a "short ton" of 2000 pounds would be equally absur

Let us suppose that the interpretation of an agreeme depends upon the meaning of the word *afternoon*. Is *afterno* "the whole time from noon to midnight" or is it "the earli part of that time as distinguished from evening"? If it is t latter, between what hours does this earlier part lie?

Abstract words such as Freedom, Liberty, Equality, Fr ternity, Democracy, Rights, Imperialism, Socialism, Capitalis

ionalism, and Education should have their meanings closely nined when they enter into discussions. They may be used sense which, if dragged from its place of concealment, we ıld instantly put from us. In one morning, a writer found different meanings of the word "imperialism" used by ten rent authors. Some of these meanings were political, some tary, some economic and some racial.

t is possible for a meaning to change in the course of a ıssion, leading to a false conclusion.

n all serious arguments, whether they be conducted in ıament, in courts of law, in the correspondence columns of ıewspapers, or in our own homes, it is of the utmost impor- e that the key words be clearly defined. Definitions are ılly essential if our thinking is to be conducted without ıusion.

A group of intelligent persons who are aware that they lack ear meaning of some important word will talk the matter until such a meaning emerges. When agreement on the ıition is reached, the discussion is likely to be fruitful. If ement on the definition cannot be attained, further dis- ıon is useless.

*Traditionally, all attempts to impose restrictions on the om of which we are proud are resisted by us.*

*The enactment of the proposed legislation would impose a iction on this freedom.*

## *Pitfall Number Seven*

Silly rooster! To imagine that, because the sun rose after crowed, it rose because you crowed! Let us see if the ter is alone in his silliness.

*"The best cure for a cold I know is 'Kil-ca-tar'. Before bottle was half finished, I felt like a new man."*

*"I think that that marriage must have worried him dreadfu She was his only child, you know. No, he never talked about but I noticed that his hair began to turn white after the weddin*

These errors in reasoning are examples of the fallacy "After, therefore because", or the fallacy of the rooster.

Very similar to the fallacy of "After, therefore because" that of "Along with, therefore because". This fallacy illustrated by the following example.

*"That Blank boy is exerting a very bad influence on our J Since they became friends, Jim has lost interest in his studies."*

If one event follows another, or if two events occur at same time, we may suspect that one is cause and the ot effect. This suspicion is a hypothesis that requires verif tion. It is a sign of immaturity, inferior intelligence, inadequate education to leap to the conclusion that a ca and effect relationship exists because two events have occur successively or simultaneously.

## *Pitfall Number Eight*

We recall the danger involved in jumping to a general clusion after considering a few instances. The error, o called the fallacy of rash generalization, is illustrated by following examples:

*"The Chinese have marvellous memories. The second tim entered Hwi Tung's laundry he said 'B 17' and handed me parcel."*

*"The police are very strict about parking in Lollipopolis. I dn't left my car more than twenty minutes in front of a hydrant en I received a ticket."*

*"They have a wonderful system of education in Ruritania. e of the Ministry officials brought me to four schools and they were above the average of ours."*

## *Pitfall Number Nine*

A teacher trying to impress upon her class the correct use 'saw' and 'seen', requested four children to come forward. ch was given a part to play. One child represented 'saw', a rdy fellow who could walk alone; another took the part of en', a sickly person unable to walk without help. The er two were 'have' and 'had', very kindly people, always dy to go to the aid of those in trouble. Without assistance, w' stalked boldly across the room, but poor 'seen' was lpless until his arm was taken by 'have' or 'had'. This ourceful teacher used an 'analogy' between certain words d certain kinds of people to make certain facts very vivid.

When interest on money is compared with rent for a use; when attention is called to the parallel between photo-nthesis and the operations of a factory; when a comparison tween an electric current and a flow of water is made; or en a preacher is likened unto a sower who "went out to sow s seed", an analogy is used to make something more easily derstood.

However, it is not with the illustrative value of analogies at we are concerned, but with their use in reasoning. The ild argued by analogy when he thought that, because snow s white like sugar, it would also be sweet. Adults should

not be too ready to smile at the child's mistake. How many of them have reasoned by analogy that the bat is a bird and that the whale is a fish?

The general plan of this kind of reasoning is as follows: If two things resemble each other in certain ways, they will also resemble each other in more ways. Thus,

*If A has the attributes m, n, p, q, etc.,*
*and*
*if B has the attributes m, n, p, etc.,*
*therefore, B has the attribute q.*

The 'etc.' in each case stands for additional attributes of A and B which were not considered in arriving at the conclusion. However, these other properties are extremely important. Some may be other attributes possessed in common by A and B, the remainder being points of difference. If one of the attributes in the 'etc.' group of B were such that B could not possess *q*, then the conclusion would be false. It would, of course, be false if it happened that *q* was one of the points of difference between A and B. It is clear that reasoning by analogy is a risky process.

A few examples will be considered. In criticizing certain changes in the British Cabinet which consisted mainly in an exchange of portfolios, a speaker was reported as having said:

"*To alter the arrangement of the cards is not to strengthen the hand.*"

There is a certain plausibility in the argument that since re-arrangement of the cards held is without effect on the value of the hand, an exchange of portfolios is ineffective in strengthening a Cabinet. However, a moment's reflection will reveal that the argument is extremely feeble. In bridge, the queen

pades does not alter in value if the cards' positions in one's d are changed, but a minister, because of special talent or rest, may be much more suited to the work of one depart- t than to that of another.

Another illustration will be given.

Parent: "*Bob finds Cosmetology very difficult. He dislikes subject so much that he has asked me to request that you nit him to drop it.*"

Teacher: "*Mrs. Blank, you do not believe that tasty and ly swallowed foods are the best for your children. As you v, there are certain foods that require chewing and are disliked he children. Some of these unpopular foods are among the t nutritious. It is the same with Cosmetology. The subject ires effort. It may not always be enjoyed. Yet we believe the demands it makes on the pupils render it a subject of the est educational value. Bob should keep on trying.*"

Mrs. Blank might have replied, "I fail to see any parallel veen food and school subjects or between eating and llectual effort. Even if I gave your analogy a limited ptance, I could use it to illustrate the value of popular jects requiring little effort. Milk is a highly nutritious l which is enjoyed by children and needs no mastication. ne of us believes that a tough beefsteak is preferable to a ler one. I fear that your analogy is unimpressive as an stration and unacceptable as an argument."

Argument by analogy is extremely common. There are ple who can produce an analogy in every argument on ch they enter, and frequently they do so with an air of ching the matter. They are encouraged by the readiness n which uncritical listeners swallow this kind of reasoning, by the reputation they gain of being practical, realistic

folk "with both feet firmly planted on the ground". Analo proves nothing. When used in reasoning, the conclusi derived from it should be regarded as a hypothesis requiri careful investigation.

## EXERCISES

In each of the arguments below, accept the assumptio *whether or not you agree with them.* In some of the argumen the conclusion is attained by incorrect reasoning; in othe the reasoning is satisfactory. Examine each argument a decide whether the reasoning is valid or invalid (fallaciou In the invalid arguments, there may be more than one falla

(*a*) Mrs. Bluthund: "The Spindrifts inherited mon recently."

Mrs. Sleuther: "Isn't that splendid! I hadn't hea about it."

Mrs. Bluthund: "They are not saying anything about but I know that her uncle died in California in the spring, a last night Tom told me that they have bought a new car an refrigerator."

(*b*) A man was requested to identify a coniferous t growing in a park. After counting the needles in a cluster, declared that the tree was a white pine. When asked explain how he identified the tree, he said, "All you have do is to count the number of needles in a cluster. White p has always five."

(*c*) Mother: "If Mabel went to the show, she did practise her music."

Father: "She was at the piano so long that I went to cellar to finish my newspaper."

Mother: "Good! She hasn't gone to the show."

(*d*) Mr. Portly: "You know, Scraggie, if you would laugh, would grow fat and handsome like me."

Mr. Scraggie: "Heaven forbid! But, what will happen if I t laugh?"

Mr. Portly: "You can't escape the logic, Scraggie. You 't grow fat and handsome."

(*e*) The Postal Department charges uniform rates for all rs carried within the country, irrespective of distance. policy has been profitable. If the railway companies to charge uniform rates, irrespective of the distance elled, they would make far more money.

(*f*) "A is without doubt a man of bad character."

'Why?"

'He is very intimate with B."

'What is the matter with B?"

'No friend of A could be straight."

(*g*) "You won't be allowed in, Bill."

'Why not?"

'It says distinctly that children accompanied by their nts or guardians will be admitted. Where are your nts?"

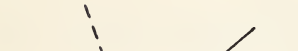
(*h*) There is nothing cruel in the training of a bear because bear enjoys the fun.

(*i*) If you saw a drowning man, you would feel it your duty ry to save him. When a poor man asks you for ten cents, have a similar obligation to give it to him.

## REVIEW EXERCISES

### 1. *Mathematical Exercises*

*A*




**1.** (*a*) What is an assum
tion?

(*b*) Using only the infor
ation conveyed by the acco
panying figure, a pupil reason
as follows:—

$$x + x + y + y = 180,$$
$$\therefore 2x + 2y = 180,$$
$$\therefore x + y = 90.$$

What assumption did he make about the lines $AO$ and $O$

**2.** A pupil was given the following problem: If the sum
the angles $A$ and $B$ is 90°, and the difference is 50°, find
size of the angle $A$. He found that $A$ was 70°. Wh
assumption was made in the solution of the problem?

**3.** State the assumption made in each of the follow
statements.

(*a*) "The area of the quadrilateral $CDEF$ is 25 squ
inches. Therefore, each side is 5 inches in length."

(*b*) "The length of one of the sides of the quadrilateral i
units. Hence, the perimeter is $4p$ units."

(*c*) "The area of the quadrilateral is $ab$ square inches, a
one side is $b$ inches long. Therefore, the length of the adjac
side is $a$ inches."

**4.** What assumptions are made in the following statemen
"If two triangles of the same area have a common base, th
are between the same parallels."

. With the aid of a figure, give the meaning of each of the ving terms:—(*a*) acute angle, (*b*) obtuse angle, (*c*) straight , (*d*) reflex angle, (*e*) complementary angles, (*f*) supple- ary angles, (*g*) adjacent angles, (*h*) vertically opposite s, (*i*) included (or contained) angle, (*j*) intersect, (*k*) versal, (*l*) alternate angles, (*m*) corresponding angles, (*n*) ior angle and an interior opposite angle of a triangle.

. With the aid of a figure, give the meaning of each of the ving terms:—(*a*) polygon, (*b*) hexagon, (*c*) pentagon, (*d*) rilateral, (*e*) parallel lines, (*f*) trapezium, (*g*) parallelo- , (*h*) rhombus, (*i*) rectangle, (*j*) square, (*k*) isosceles gle, (*l*) equilateral triangle, (*m*) right triangle, (*n*) isosceles triangle.

With the aid of a figure, give the meaning of each of ollowing terms:—(*a*) congruent triangles, (*b*) altitude, (*c*) mscribe, (*d*) diagonal, (*e*) hypotenuse, (*f*) similar triangles.

With the aid of examples, explain (*a*) converse, (*b*) apositive, (*c*) ratio, (*d*) proportion.

What is wrong in each of these definitions? (*a*) If two ght lines do not meet however far they are produced in r direction, they are said to be parallel.

) A rectangle is a parallelogram in which each angle is a angle.

) Similar triangles are triangles in which the angles are ctively equal, and in which the sides including the equal s are proportional.

) A right angle is an angle which is greater than an acute and less than an obtuse angle.

) An equilateral triangle is a triangle containing three of equal sides.

0. Make a list of 10 angles (measured in degrees) which are able to construct with ruler and compasses.

**11.** One angle in a right triangle is 22°. What is the s of the third angle?

**12.** In a certain right triangle, the two acute angles a equal. What is the size of each?

**13.** The angles of a triangle are $(2x + 30)°$, $(5x - 7($ and $(3x + 10)°$. Find the size of each angle.

**14.** In a right triangle, one of the acute angles is five tir the other. What is the size of each?

**15.** In an isosceles triangle, each of the equal angles is t and a half times the third angle. Find the size of each of three angles.

**16.** An angle and its supplement are vertically oppos angles. What is the size of the angle?

**17.** (*a*) If two triangles are congruent, seven magnitu of the one are respectively equal to seven of the other. W seven are they?

(*b*) In order to construct a triangle which is congru with a given triangle is it essential to know the value of e of these seven features?

(*c*) What three things is it sufficient to know abou triangle in order to construct a triangle congruent with (You should be able to give three correct answers.)

**18.** (*a*) Give the converse of the following statement: the sides of one triangle are respectively equal to the side another triangle, the angles of the one triangle are respectiv equal to the angles of the other.

(*b*) Is the original statement true? Is its converse tru

**19.** The statue of a man 6 feet tall is 15 feet in height.

(*a*) What is the ratio of the height of the statue to height of the man?

*b*) If the nose of the man is $2\frac{1}{8}$ inches long, what is the th of the nose on the statue?

*c*) If the middle finger on the left hand of the statue is $9\frac{3}{8}$ es long, what is the length of the corresponding finger of nan?

0. Construct a $\triangle ABC$ in which $AB = 3.2$ in., $AC = 2.1$ $\angle A = 75°$. From $A$ draw a perpendicular to $BC$. sure and record the length of (*a*) $BC$, (*b*) the perpendicular. not use protractor or set squares.)

1. Construct a $\triangle ABC$ in which $AB = 7.2$ cm., $BC =$ ı., $AC = 5.4$ cm. By means of compasses and straight , find the midpoint $M$ of $BC$. Join $AM$. Measure and rd the size of (*a*) $\angle BAC$, (*b*) $MA$, (*c*) $MB$, (*d*) $MC$.

2. Without using a protractor, construct a $\triangle ABC$ in h $BC = 2\frac{1}{2}$ in., $\angle A = 60°$, $\angle C = 30°$. Bisect $\angle$s $A$ $C$. Let the bisectors meet at $O$. Measure and record the of (*a*) $\angle B$, (*b*) $\angle AOC$.

3. Without using a protractor construct two different ıgles $A_1BC$ and $A_2BC$ each fulfilling the following con- ›ns: $A_1B = A_2B = 5.1$ cm., $BC = 6.8$ cm., $\angle C = 45°$. sure (*a*) $A_1C$, (*b*) $A_2C$, (*c*) $\angle BA_1C$, (*d*) $\angle BA_2C$.

4. Without using a protractor construct a quadrilateral $CD$ in which $AB = 2.1$ in., $BC = 2.7$ in., $CD = 2.3$ in., $= 2.9$ in., $\angle B = 75°$. Make the constructions and surements necessary in finding the area of the quadrilateral. ıt is the area of $ABCD$?

5. Without using a protractor, construct a square on a 5 cm. long. Measure the diagonals.

6. Construct a square with a diagonal of 2.8 inches.

7. Without using a protractor, construct a rectangle $CD$ in which $AB = 2.5$ in. and $BC = 2.1$ in. On $BC$ ›ase, construct a triangle equal in area to the rectangle. ıt is the altitude of this triangle?

**28.** Without using a protractor, construct a rhombus $AB$ in which $AB = 5.2$ cm., and $\angle B = 135°$. Let the diago intersect at $O$. Measure (*a*) $OA$, (*b*) $OC$, (*c*) $OD$, (*d*) $OB$.

**29.** The diagonals of a rhombus $ABCD$ intersect at $AO = 2.7$ cm., $BO = 2.1$ cm. Construct the rhombus measure its sides.

**30.** Without using a protractor, construct a parallelog $ABCD$ in which $AB = 2.8$ in., $BC = 1.9$ in., $\angle B = 6$ Measure and record the length of (*a*) $AC$, (*b*) $BD$.

**31.** Find the area of a triangle in which the base is $x$ units and the altitude $2x + 6$ units.

**32.** Find the area of a square of which the side is $x$ units in length.

**33.** The area of a square is $x^2 + 16x + 64$ square un Find the side.

**34.** The area of a square is $x^2 - 10x + 25$ square un Find the side.

**35.** Find the area of a rectangle $x + 7$ units in length $x - 6$ units in width.

**36.** The area of a rectangle is $x^2 - x - 6$ square un and its length is $x + 2$ units. Find the width.

**37.** The two parallel sides of a trapezium are $x$ units an units in length, and the perpendicular distance between th is $d$ units. Give the area of the trapezium.

**38.** The two parallel sides of a trapezium are $n$ units an units in length. The perpendicular distance between ther $m$ units. Give the area of the trapezium. What other na may be given to the figure?

**39.** The two parallel sides of a trapezium are $s$ and $s$ u in length. The perpendicular distance between them i

ts. Give the area of the trapezium. What other name y be given to this figure?

**40.** The two parallel sides of a trapezium are $x - y$ and $y$ units in length. The perpendicular distance between n is $x$ units. What is the area of the trapezium?

**41.** The two parallel sides of a trapezium are $x + 4$ and 8 units in length. The perpendicular distance between n is $x + 7$ units. What is the area of the trapezium?

**42.** A pilot climbing at an angle of 45° finds that in a short e he is vertically above a house which is three-quarters of a distant from his starting point. What is his altitude?

**43.** An aeroplane flew west from point $A$ at 180 miles per r for an hour and a third. Then it flew south at 160 miles hour for two hours to a point $C$. If it had flown directly n $A$ to $C$ at 150 miles per hour, how long would the journey e taken? (Assume that there is no wind.)

**44.** The following lengths are the sides of right triangles:

3, 4, 5 units
6, 8, 10 units
9, 12, 15 units
12, 16, 20 units

. . .
. . .

This series of lengths could be continued indefinitely. e the two sets which should immediately follow the 12, 16, set. Test your answers by using the converse of the hagorean theorem.

*B*

**5.** State a theorem regarding the exterior angle formed by ucing one side of (*a*) a rectangle, (*b*) a parallelogram.

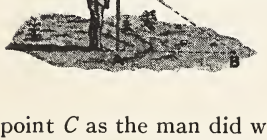
**46.** $AB$ and $CD$ are two lines intersecting at $O$ such th $AO = OB$ and $CO = OD$. Show (*a*) that $AC = BD$, that $AC \parallel BD$.

**47.** The illustrati shows an old device measuring a distance indirectly. A man pul down the brim of his until he could look alo its edge to the inaccessi point $B$. Then he turned around and, looking past the edge of the brim, observed a point $C$ which was accessible. He measured $AC$ and took the length as that of $AB$.

(*a*) What principle of congruence was used?

(*b*) What must be assumed about the levels of $B$, $A$ $C$? Why?

**48.** A variation of the preceding device will be describ




The short cross-stick is atta ed to the vertical staff so tha is movable. The cross-piec adjusted until the observer look along it and see the point Then, keeping the staff verti he turns it around and locate point $C$ as the man did with the brim of his hat.

How would the measurement be affected if the staff w not quite vertical when it was turned?

49. The angles of a triangle are $k°$, $2n°$ and $k°$. Show that
s the complement of $n°$.

50. The diagonal $BD$ of the quadrilateral $ABCD$ bisects
at right angles. Prove ($a$) that $AB = BC$, ($b$) that
$= DA$.

51. In $\triangle ABC$, $AB = AC$, and $D$, $E$, and $F$ are the
dle points of $BC$, $CA$ and $AB$ respectively. Prove that
$= DF$.

52. $AC$ is a diagonal of the quadrilateral $ABCD$. $\angle BAC$
$\angle ACD$ and $\angle DAC = \angle ACB$. Show that $ABCD$ is a
llelogram.

53. Show that a straight line perpendicular to one of two
llel straight lines is also perpendicular to the other.

54. Find without measurement the sum of the interior
es of a hexagon.

55. The sum of the interior angles of a polygon is 10 right
es. How many sides has the polygon?

56. Find without measurement the sum of the exterior
es of ($a$) a triangle, ($b$) a quadrilateral, ($c$) a pentagon, and
a hexagon.

57. How many sides has a regular polygon if one of the
rior angles is ($a$) 90°, ($b$) 60°, ($c$) 1°, ($d$) $n°$?

58. Is it possible to have a regular polygon one of the
rior angles of which is 7°? Justify your answer.

59. $ABCD$ is a square, and $E$ is the middle point of $AB$.
ve that $ED = EC$.

60. $D$ is the middle point of the side $AB$ of $\triangle ABC$.
is produced to $E$ so that $DE = DC$. Prove that $BCAE$
parallelogram.

61. Show that each of the angles opposite the equal sides
a isosceles triangle is acute.

**62.** In $\triangle ABC$, $AB = AC$, and $OB$ and $OC$ are t bisectors of $\angle$s $ABC$ and $ACB$. Prove that $OA$ bise $\angle BAC$.

**63.** A 6-inch rule is placed obliquely on a sheet of ru paper so that the ends of the rule lie on the fourth and tw tieth lines of the sheet. Find the distance on the rule betw the eighth line and the thirteenth.

**64.** Prove that the three altitudes of an equilateral trian are equal.

**65.** Replace each of the following phrases by a single wo with the same meaning:—(*a*) equilateral parallelogram; rectangular rhombus; (*c*) quadrilateral with two parallel sid (*d*) quadrilateral with two pairs of parallel sides.

**66.** (*a*) Three lines are drawn parallel respectively to three sides of an equilateral triangle. What kind of fig do they form? Give your reasons.

(*b*) Four lines are drawn parallel respectively to the f sides of a square. What kind of figure do they form? G your reasons.

**67.** Two boys on a road which runs north and so observed a distant tree in a field. John found that its direct from him was exactly southeast. Jim found that its direct from him was northeast. What geometric fact could t deduce? Give your reasons.

**68.** Show how to divide a given triangle into (*a*) four sim triangles, (*b*) nine similar triangles.

**69.** An isosceles right triangle has each equal side of l length. A second isosceles right triangle is formed in w one of the equal sides is the hypotenuse of the first. A t triangle is formed similarly from the second, and a fou similarly from the third. Make the construction and find length of the hypotenuse of the fourth triangle.

70. Two boys, one directly north of the other, watched a travelling north on a road. When it was nearly west of ve, he said, "I would like to own a patch of ground as big rom me to you to the car." (Dave had not learned about ngles and vertices.) When the car was almost out of sight, said, "You wouldn't be satisfied with as much land as e is now, me to you to the car." What proposition had still to learn?

71. A diagonal of a parallelogram bisects each of the es through which it passes. What name can be applied his parallelogram? Give your reasons.

72. $ABCD$ and $ABEF$ are parallelograms. Show that $EF$ is also a parallelogram. Are there any exceptions?

73. $ABC$ is a triangle in which $AB = AC$. The perpendular from $A$ to $BC$ meets $BC$ in $D$. If you are given the ths of $BC$ and $AD$, how would you construct the triangle?

74. Given the altitude, construct an equilateral triangle.

75. Given the vertical angle and the altitude of an isosceles ngle, construct the triangle.

76. Given the hypotenuse and one of the other sides, truct a right triangle.

77. The side $BC$ of a triangle $ABC$ is 2 inches in length. angle $B$ is 75°. $A$ lies on a line parallel to $BC$ and 3 inches nt from it. Construct the triangle.

78. Given two sides of a triangle and the median drawn to of these sides, construct the triangle.

79. Given the diagonals of a parallelogram and the angle ained by them, construct the parallelogram.

80. Given the diagonals, construct a rhombus.

**81.** $ABCD$ is a parallelogram. $E$ is the middle point $AB$, and $F$ is the middle point of $CD$. Show that $DE$, $E$ and $FB$ divide the parallelogram into four congruent triangl

**82.** $ABC$ is an equilateral triangle. $AD$ is drawn perpe dicular, and equal, to $AB$. $C$ and $D$ are on opposite sides $AB$. Find, without measurement, the size of the angle $DB$

**83.** If two straight lines intersect, the bisectors of t four angles are two perpendicular straight lines.

**84.** Construct a rectangle $ABCD$ in which $AB = 6$ c and $BC = 5$ cm. On $AB$ as base, construct a rhombus eq to the rectangle.

**85.** With straight edge and compasses, construct a paral ogram $ABCD$ in which $AB = 3$ in., $AD = 2.5$ in., and angle $DAB = 67\frac{1}{2}°$. Construct a parallelogram $ABEF$ eq to $ABCD$, in which the angle $FAB = 45°$.

**86.** $ABCD$ is a quadrilateral in which the diagonal $D$ bisects the diagonal $AC$. Show that $DB$ divides the qu rilateral into two equal triangles.

**87.** $P$ is a point on the median $AD$ of a triangle $AB$ Prove that the triangles $ABP$ and $ACP$ are equal.

**88.** Prove that the square on the diagonal of a given squ is double the original square.

**89.** Prove that the sum of the squares on the diagonals o rhombus is four times the square on one of the sides.

**90.** (*a*) Give the converse of the following statement: two straight lines are parallel, they do not meet however they are produced in either direction.

(*b*) Can this converse be proved?

91. (*a*) Give the contrapositive of the following statement: ıe sides $AB$ and $AC$ of a triangle $ABC$ are equal, then the les $B$ and $C$ are equal.

(*b*) If the statement in (*a*) has been proved, does the trapositive require proof?

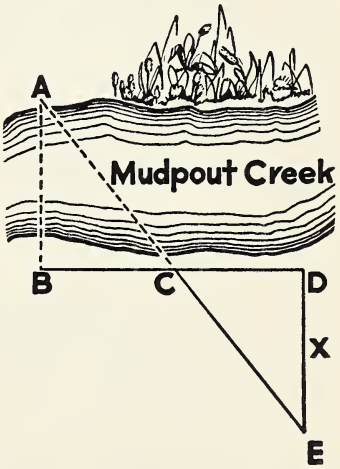
*C*

92. A square room of side 18 feet is 12 feet high, these surements being correct to the nearest inch. A wire is tched from one corner of the floor to the centre of the ng. What is the length of the wire to the nearest inch?

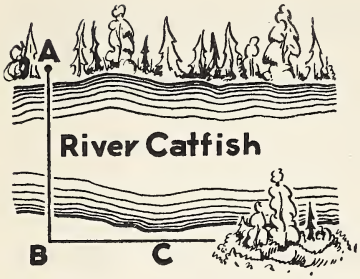
93. Ralph, who was standing at the point $B$ wished to the distance from $B$ to $A$.

He marked out a line pendicular to $BA$ and on ook a point $C$. Then he d a point $D$ on the line that $CD$ was equal to Next, he walked along e $DX$ at right angles to until he came to a point line with $C$ and $A$. He sured $DE$ and took the th as being equal to that $A$.

He gave a satisfactory anation of his procedure. What explanation would give?

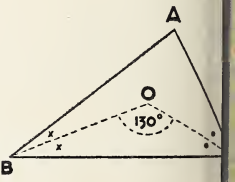



94. Margaret thought she would use Ralph's procedure in r to find the width of the lordly River Catfish.




Everything went we until she came to *C*. The she realized that a clum of trees prevented h from completing the pla But she was a resourcef girl. She found the wid of the Catfish by a mo fication of Ralph's pla Instead of using congrue triangles, she used similar triangles. How did she obtain t distance from *B* to *A*?

**95.** *BO* and *CO* are respectively the bisectors of the angles *ABC* and *ACB*. Calculate the size of the angle *A*.



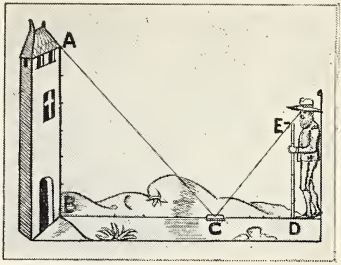

**96.** The following are the lengths of the sides of right t angles:

8, 15, 17 units
12, 35, 37 units
16, 63, 65 units
20, 99, 101 units
24, 143, 145 units

. . .
. . .
. . .

This series could be continued indefinitely. Give the that immediately follows 24, 143, 145. Test the accuracy your answer.

**97.** The illustration ows an interesting de- e once used in finding ghts. The observer ob- ned from the mirror at he reflection of the top he tower. It is known t when reflection occurs angle $ACB$ is equal to angle $ECD$. Explain v the height $AB$ is cal- ated.

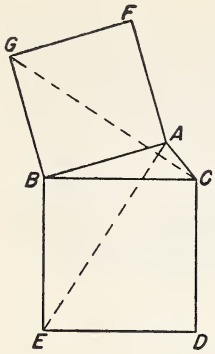



**98.** Two triangles have bases in the ratio of $m$ to $n$, and tudes in the ratio of $n$ to $m$. Compare their areas.

**99.** Some children hung their swing from the straight but sloping limb of a tree. What shape has the closed figure? Will it continue to have this shape during the swing? Draw a figure to show its approximate shape when it has swung through an angle of 30° from its original vertical position.

**100.** Stanley made a parallelogram with four strips of dboard which he fastened at the vertices with thumbtacks hat the sides were free to move. Holding one side fixed, rotated one of the free corners. Did the figure remain a allelogram? Why? In what position is its area greatest?

**101.** A parallel to the side $BC$ of a triangle $ABC$ cuts $AB$ . $AC$ in $D$ and $E$ respectively. Show that the triangles $E$ and $ACD$ are equal.




**102.** In the accompanying figur $BAFG$ and $BEDC$ are squares drawn c two sides of any triangle $ABC$. Prov that the triangles $ABE$ and $GBC$ a congruent.

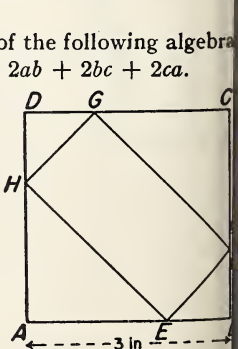
**103.** A triangle has sides of th following lengths: 21 in., 20 in., 29 i Without drawing the triangle, find i area.

**104.** A road has an upward slope of 20° to the horizonta A tunnel, bored directly under the road, slopes downwar making an angle of 10° with the horizontal. By means of scale drawing, find the depth below the road of a point in th tunnel 100 feet from the mouth.

**105.** $ABC$ is a triangle in which the angle $C$ is a right angl $D$ is any point in $BC$. Show that $AB^2 + CD^2 = AD^2$ $BC^2$.

**106.** Give a geometric illustration of the following algebra fact: $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$.

**107.** In the accompanying figure, $ABCD$ is a square, and $BE = BF = DG = DH = 1$ inch. (*a*) Show that $EFGH$ is a rectangle. (*b*) Calculate (*i*) the perimeter of $EFGH$, (*ii*) the length of a diagonal of $EFGH$.




**108.** Prove that a rhombus is formed when the middle points of adjacent sides of a rectangle are joined. What is the ratio he area of the rhombus to that of the rectangle?

**109.** $ABC$ is a triangle in which the angle $A$ is a right gle. $D$ is the middle point of $BC$. Prove that $BD = DA$ $DC$.

**110.** If $\frac{2.9}{3.5} = \frac{2.3}{x}$, find $x$ (*i*) by calculation, (*ii*) by means a geometric construction.

**111.** In the triangles $ABC$ and $DEF$, $\angle B = \angle E$ and $= \angle F$. $AB = 3$ in., $BC = 4$ in., $CA = 5$ in., $DE =$ in. Calculate the length of (*i*) $EF$, (*ii*) $DF$.

**112.** Prove that, in an equilateral triangle, the sum of the ares on the three sides is four times the square on the pendicular from a vertex to the side opposite.

**113.** The sides of a triangle are in the ratio 3 : 4 : 5 and area is 294 square centimetres. Calculate its perimeter.

## 2. *Non-Mathematical Exercises*

**1.** What is meant by an ambiguity?

**2.** Point out an ambiguity in each of the following statements.

(*a*) Newfoundland tars land. (Headline in newspaper.)

(*b*) I say that you Æcus the Romans are able to conquer.

(*c*) She ate her ice-cream quickly because it was hot.

(*d*) When four years old, my grandmother died.

(*e*) I did say he was a liar and I am sorry for it.

**3.** State a rule broken in the following 'definitions':

"What is a beetle?"

"A beetle is a Coleopterous insect."

"What do you mean by the term 'Coleoptera'?"

"It is the order of insects which contains the beetles."

4. Criticize the following definitions:

(*a*) A chair is an article of furniture on which to sit.

(*b*) A horse is a beast of burden.

(*c*) A conservative person is one who favours conservatism

5. (*a*) Give the converse of the advertisement *If it's Fontaine, it's a good pen.*

(*b*) If the original statement is accepted, must the converse be accepted also?

6. What is assumed in each of the following statements:

(*a*) Provided that your services are satisfactory, you may expect a reasonable rise in salary next year.

(*b*) Is this soap good! Why, we sell more of it than of any other kind.

(*c*) You need have no hesitation in purchasing nationally advertised articles.

7. The chief of police in a certain city found that in 70 per cent of the automobile accidents in a year the drivers were men. He concluded that women were better drivers than men.

What assumptions entered into his argument?

What additional facts would he need before a comparison could be made?

8. (*a*) Give an example of the use of indirect proof in everyday reasoning.

(*b*) To what hazard is the use of indirect proof exposed?

9. Teachers advise their pupils to draw general geometric figures unless special cases are demanded by the enunciation. What danger is present when special figures are used unnecessarily?

**10.** Criticize the following argument in favour of making wings in science classes:

If the student's observations are inaccurate, then his wing is inaccurate. Hence, inaccuracy in the drawing vides evidence of inaccurate observation.

**11.** (*a*) What is the contrapositive of *If it is bad for the rm, then it is bad for the bee*?

(*b*) If the original statement is accepted, must the con- positive be accepted also?

**12.** Suppose that the following statement is true: *If Tom sed his examination, he was given a bicycle.*

(*a*) If you know that Tom was given a bicycle, would you that the following statement is true, false, or uncertain? *n passed his examination.*

(*b*) If you know that Tom was not given a bicycle, would say that the following statement is true, false, or uncer- ? *Tom failed his examination.*

(*c*) If you know that Tom did not pass his examination, ld you say that the following statement is true, false, or ertain? *Tom was not given a bicycle.*

**13.** Assume the truth of the following assertion only: *If ubstance contains starch, then it will give a blue coloration solution of iodine.*

(*a*) Are you justified in asserting that a substance that does contain starch will not give a blue coloration with iodine tion?

(*b*) Are you justified in concluding that a substance con- s starch if it gives a blue coloration with iodine solution?

(*c*) Are you justified in concluding that a substance does contain starch if it fails to give a blue coloration with ne solution?

**14.** Criticize the following argument:

"This year our hockey team has lost four of the five game played. Obviously, the new coach deserves the blame because in the last two years we lost only two games altogether."

**15.** "Our new neighbour is a poor housekeeper because this morning I dropped into her kitchen for a few minutes and her dishes were still unwashed."

Is this argument a good one? If you regard it as poor give your reason.

**16.** A pupil found that the pressure of the air would support columns of water of the following heights: six inches; eleven inches; eighteen inches; two feet; twenty-seven inches; two and a half feet.

He concluded that the pressure of the atmosphere would support a column of water of any height. Criticize the conclusion.

**17.** Try to frame a definition of the word 'street'. (Questions such as the following should be considered. Must a street be open to the public? Can you have a street without any houses? If there must be houses, is it necessary that they should be on both sides? If there must be houses, how far apart may they be? May a street have a 'dead-end'? May it have any length, great or little? If a bridge forms a connecting link between two portions of a street, is it part of the street?)

**18.** Assume the truth of the following statement: *If the Anopheles mosquito is absent, there is no malaria.*

If you were given the following three pieces of information what conclusions, if any, would you be justified in drawing?

(*a*) There is malaria in Pongorena.

(*b*) The Anopheles mosquito is found in Bwaana.

(*c*) There is no malaria in Koiba.

**19.** Assume the truth of the following statements: *If logy is used in reasoning, the conclusion derived from it uld be regarded as a hypothesis which must be tested. Mr. ith's conclusion is a hypothesis which must be tested.*

Criticize the following inference: Mr. Smith reasoned by logy.

**20.** Three men were engaged in a conversation.

A began the conversation and said either, "I am a Noble" "I am a Hunter." We do not know which statement he de.

B said to A, "You said you were a Hunter."

B said to C, "You are a Noble."

C said to A, "You are a Noble."

From the above conversation, decide whether each is oble or a Hunter. You have two theorems to guide you:

Theorem 1. A Noble always tells the truth.

Theorem 2. A Hunter always lies.[1]

State a third necessary assumption.

**21.** What fallacy is illustrated in the following quotation n *Punch*?

Tourist (pointing to molehills): "What are these?"

Giles: "Them's oompty-toompties."

Tourist: "And what are umpty-tumpties?"

Giles: "Oompties wot the toompties make."

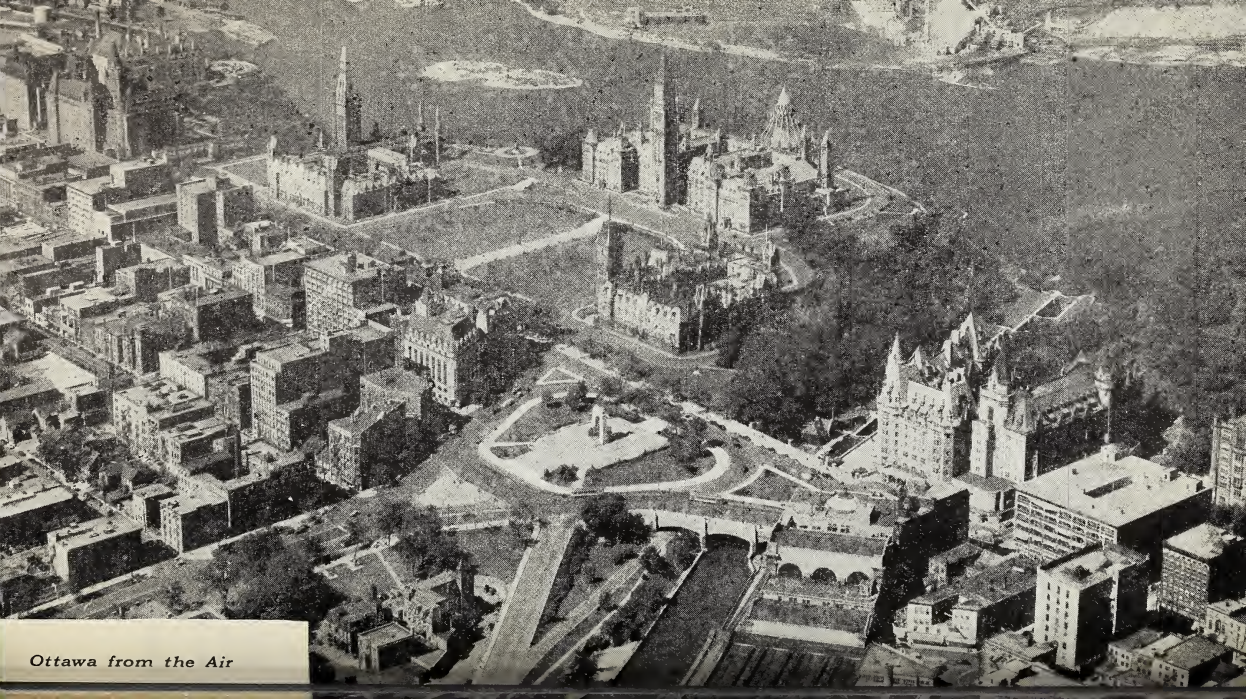
Tourist: "But what are tumpties?"

Giles: "Whoy, wot makes the oompties, you fool."

**22.** Criticize the following argument:

Under the Halcyon government, there were many bankcies and much unemployment. No man with any respect his job would cast his vote for a Halcyon candidate.

[1] Nyberg, J. A., *School Science and Mathematics*, June 1933.



*Ottawa from the Air*

CHAPTER XIV

# AREAS AND VOLUMES

*"For he by geometric scale*
*Could take the size of pots of ale."*

Samuel Butler

## *A Bird's Eye View*

The air view of the City of Ottawa on page 266 is worth e than a glance. What an abundance of geometrical n! Parallel lines, rectangles in thousands, arcs of circles, ngles, rectangular solids, cylinders, cones, prisms,—you find them all in the picture. The builders of our cities e constant use of the common figures and solids of ge- try. Notice the frequency with which the rectangle urs; it has been called the basic form in building. The amid and the cone appear in a multitude of spires, steeples, towers.

The small towns, villages, and farming communities of ada also offer many examples of the use of geometry. of the commonest and simplest kinds of building is a angular solid with a prism on top (see Fig. a).

You will see this form in many ses and barns. Silos and water ks are generally cylindrical, the nder sometimes being surmount- y a cone. You may have seen erical tanks used for storing oline (see page 277); such tanks e the advantage of requiring material for a given volume.

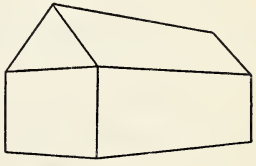
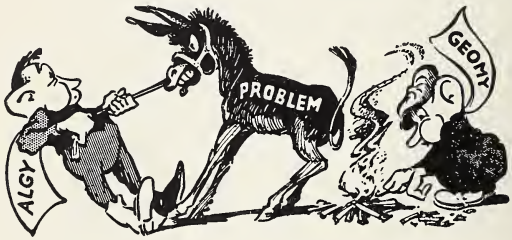


Fig. a

ny church steeples have the form of a pyramid or of a cone.

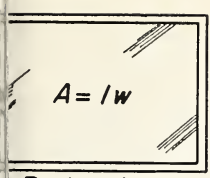
Now the building and upkeep of structures such as the described give rise to questions which require mathemati for their answers. How big is it? How much material w it take? How much will it hold? What will it cost to pai it? Such questions are readily answered by a combination algebra with geometry. The formula is the key. Applyi the methods of algebra to the forms of geometry introduc us to one of the useful aspects of mathematics; each part of t subject re-enforces the other.




## *Something to Recall*

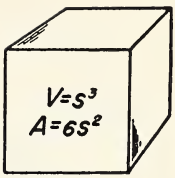
Before we set about finding formulas for new solids, let burnish a few ideas of the first course. Here is a short su mary of facts which you have met before.

An ***area*** may be described as an amount of surface; it measured in square units, such as the square inch or the squ centimetre. Likewise, the content of a solid is its ***volum*** which is measured in cubic units, such as the cubic foot or t cubic centimetre.

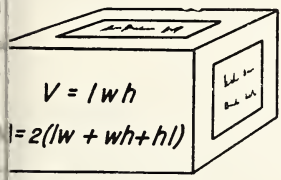



Rectangle

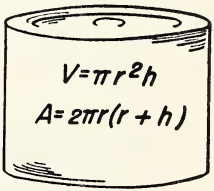
Circle




Cube




ectangular Solid




Cylinder

*ample.* Find the formula for the volume of a pipe $h$ units g with outer and inner radii of $R$ and $r$, respectively. e this formula to obtain the volume of the pipe shown in the ıre on page 270.

The volume of a pipe is the difference between the volumes the two cylinders contained by its outer and inner surfaces. erefore,

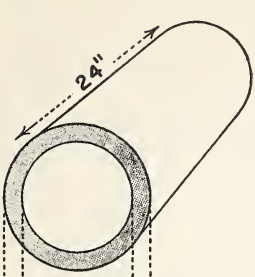
$$V = \pi R^2 h - \pi r^2 h,$$
$$= \pi h\,(R^2 - r^2).$$

An approximation to the value of $\pi$ is $3\frac{1}{7}$ which, in decimal notation, orrect to 3 figures, 3.14. A more precise approximation is 3.1416.

e third factor of this formula involves the difference of the ıares of two numbers. We met this before in connection h the theorem of Pythagoras. The difference of the ıares of two numbers can be factored thus:

$$R^2 - r^2 = (R + r)\,(R - r).$$

e factored form offers some advantage in calculation.




Thus for the pipe shown, we ha

$$V = 24\pi\,(6.5^2 - 4.5^2),$$
$$= 24\pi\,(6.5 + 4.5)\,(6.5 - 4.5),$$
$$= 24\pi\,(11)\,(2) = 528\pi,$$
$$= 528\ (22/7) = 1700,$$

correct to 2 figures.

Hence the volume of the pipe about 1700 cu. in.

Compare the calculations with those involved when t difference of squares is not factored.

### *A Reminder About Approximate Computation*

*In Book I, we noted that all numbers obtained from measurements are subject to error. For example, when we report that a fence is* 82 *yards long, we mean (in the absence of other information) that the fence is nearer to* 82 *yards in length than to* 81 *yards or* 83 *yards. The length of the fence lies between* 81.5 *and* 82.5 *yards. Similarly, if the area of a sheet of tin is reported as* 32.5 *sq. ft., we mean that its area is between* 32.45 *and* 32.55 *sq. ft.* ***Each measurement may be in doubt by one-half of the smallest unit reported.*** *Thus a measurement of* 82.5 *units may be in error by one-half of one-tenth or* 0.05 *unit;* 82.50 *units may be in error by one-half of one-hundredth or* 0.005 *unit.*

*Accuracy of a measured number depends on the number of significant digits reported; the more significant digits, the more accurate the number. In calculating areas and volumes with measured lengths, it is important to realize that* ***the answer cannot be more accurate than the least accurate of the measured lengths.*** *Therefore we round off the answer to the same number of figures as that of the least accurate number in the data. The product of* 32.7

*nd* 76.25 *should be rounded to* 3 *figures; the quotient of* 64.8 *nd* 7.8 *should be rounded to* 2 *figures.*

*When approximate numbers are added or subtracted, it s the position of the decimal point, rather than the number f significant digits, that determines the amount of rounding ff in the answer. Thus, for addition and subtraction, the approximate lengths 8.273" and 4.5" should be rounded off to ie same unit, as follows:*

| Add: | 8.3″ | Subtract: | 8.3″ |
|---|---|---|---|
| | 4.5″ | | 4.5″ |
| | 12.8″ | | 3.8″ |

*Sometimes we shall assume exact data in a problem or purposes of practice. Under this assumption, the roblem becomes an ideal one and we can get as accurate an nswer as we wish.*

## EXERCISES

*nless stated otherwise, measurements are approximate and wers should be reported to the accuracy justified by the data.)*

1. Find the area of a tablecloth 8.4 feet long and 6.2 feet e.

2. The height of a cylindrical oil drum is 48.5 inches and radius of its top is 15.2 inches. Find (*a*) the volume, (*b*) area of the drum. (Take $\pi = 22/7$)

3. A rectangular coal bin 4.4 feet high has a square base ose sides are 8.0 feet. If one ton of coal occupies 31 cu. find the number of tons the bin can hold.

4. A cylindrical silo has a diameter of 14 feet and height, feet. Find the area of its lateral surface. If a gallon of nt covers 220 sq. ft., find the amount of paint necessary to e the side of the silo two coats.

5. A silo, properly filled, holds one ton of silage for every 50 ft. of space. How many tons of silage can the silo of 4 hold?

6. Find the volume of a pipe like that shown in the examp on page 270, given the following data: inner radius, 7.5 i outer radius, 9.5 in.; length, 36 in.

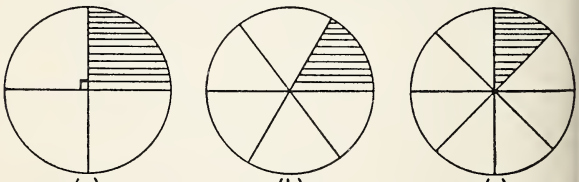
7. A circular flower bed with diameter 21 feet is surround by a gravel path 3.0 feet wide. Find the area of the gra path.

8. What length of border would be needed for both sid of the gravel path in Ex. 7?

9. Water pressure for a certain community is maintain by a cylindrical standpipe 28.0 feet in diameter and 10 feet high. If one cu. ft. of water contains 6.23 gallons, fi correct to 3 figures, the number of gallons that the tank c hold.

10. It is desired to make a reservoir by removing a rectan lar solid of earth and cementing the bottom and sides of t opening. If the bottom of the reservoir is to be 12.8 feet 10.5 feet, find the depth necessary for a capacity of 1000 cu.

### *Area of a Sector of a Circle*




Each of the shaded regions in the foregoing figures is cal a ***sector*** of a circle. Such sectors are enclosed by two ra and an arc of the circle. In order to find the area of a sect you should note the way in which the size of the sector depe on the angle between its radii, or on the length of its arc.

In Fig. a, you see that the angle of the sector is 90°, wh is 90/360 or $\frac{1}{4}$ of the whole angle at the centre of the cir Also the arc of the sector is $\frac{1}{4}$ of the circumference of the cir

reover, the area of the sector is evidently $\frac{1}{4}$ of the area of the le. Similarly, in Figs. b and c, we note that the following tions are equal:

$$\frac{\text{area of sector}}{\text{area of circle}} = \frac{\text{number of degrees in angle of sector}}{360} = \frac{\text{length of arc of sector}}{\text{length of circumference}}.$$

You will therefore not be surprised to learn that the following can be proved:

If the angle of sector is $x°$, then its area is given by

$$A = \frac{x}{360} \text{ (area of circle)} = \frac{x}{360} (\pi r^2).$$

, if the arc of the sector is $a$ units, then its area is given by

$$A = \frac{\text{arc}}{\text{cir'ce}} (\pi r^2) = \frac{a}{2\pi r} (\pi r^2) = \tfrac{1}{2} ar.$$

*mple.* The radius of a circle is 7 inches. Find the area sector of this circle given that (*a*) the angle of the sector )°, (*b*) the arc of the sector is 4 inches. ($\pi = 22/7$)

(*a*) We have

$$A = \frac{x}{360} (\pi r^2) = \frac{20}{360} \cdot \frac{22}{7} \cdot 49 = \frac{77}{9} = 8\tfrac{5}{9}.$$

Hence the area is about 8.6 sq. in.

(*b*) Since the arc of the sector is known, we have

$$A = \tfrac{1}{2} ar = \tfrac{1}{2} \cdot 4 \cdot 7 = 14.$$

Therefore the area is 14 sq. in.

### EXERCISES

1. What fraction of the area of a circle is a sector wh angle is (*a*) 60°, (*b*) 45°, (*c*) 10°, (*d*) 120°, (*e*) 180°, (*f*) 72°?

2. What fraction of the area of a circle is a sector wh arc is (*a*) one-quarter of the circumference, (*b*) two-sevent of the circumference, (*c*) one-tenth of the circumference?

3. If the radius of a circle is 7 cm., find the area of a sect whose angle is 15°. (Assume data exact.)

4. If the radius of a circle is 7 ft., find the area of a sect whose arc is 7 feet. (Assume data exact.)

5. The radius of a circle is 5.6 inches. Find the area of sector whose (*a*) angle is 36°, (*b*) arc is 3.2 inches. (Assu 2-figure data.)

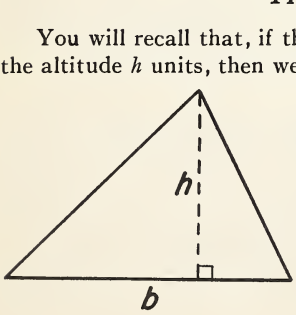
6. The radius of a circle is 25.2 ft. Find the area of a sect having (*a*) an angle of 21.6°, (*b*) an arc 42.8 ft. long.

7. In the circle of Ex. 3, find the angle of a sector wh area is 15.4 sq. cm. Find also the arc of a sector whose ar is 21 sq. cm.

8. In the circle of Ex. 5, find (*a*) the angle of a sector wh area is 6.16 sq. in., (*b*) the arc of a sector whose area is sq. in.

## *Triangles*

You will recall that, if the base of a triangle is $b$ units a the altitude $h$ units, then we have




***Area of a Triangle***

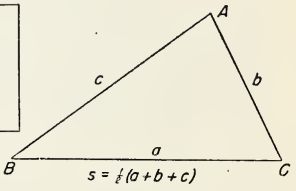
$$A = \tfrac{1}{2}\, bh$$

ere are other formulas for the area of a triangle and these proved in more advanced books on mathematics. One hese formulas is of particular interest, and it is suggested t you accept it on faith. If you are curious about its of, consult Book III of this series.

The formula to which we refer is called the "s" formula or on's formula. Heron of Alexandria proved it about . 50. If we denote the semi-perimeter of a triangle by $s$, l the three sides by $a$, $b$, and $c$, respectively, then we have

***Heron's Formula***

$$A = \sqrt{s\,(s-a)\,(s-b)\,(s-c)}$$




*ample* 1. Find the area of a triangle with sides 6, 8, and 10 ts respectively.

We have $s = \frac{1}{2}(6 + 8 + 10) = 12$. Therefore, $s - a = 12 - 6 = 6$, $s - b = 4$, and $s - c = 2$.

Heron's Formula gives

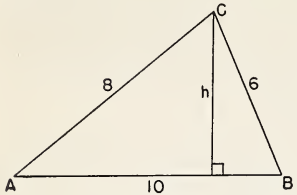
$$A = \sqrt{s\,(s-a)\,(s-b)\,(s-c)} = \sqrt{12\,(6)\,(4)\,(2)}$$

$$= \sqrt{576} = 24.$$

Hence the area is 24 square units.

With the aid of Heron's Formula and $A = \frac{1}{2}bh$, we can d an altitude of a triangle when three of its sides are known. ıdy the following example.

*Example* 2. In the triangle of Example 1, find the length
the altitude drawn from
vertex to the side 10 units lon




Using Heron's Formula w
found the area to be 24 squa
units. We also have

$$A = \tfrac{1}{2}bh = \tfrac{1}{2}(10)\,h = 5h.$$

Since the areas are the same, we get

$$5h = 24, \text{ whence } h = 24/5 = 4.8.$$

The altitude required is 4.8 units long.

## EXERCISES

Use Heron's Formula to find the area of a triangle with t
following sides:

1. 3 in., 4 in., 5 in.
2. 5 ft., 12 ft., 13 ft.
3. 13 yd., 14 yd., 15 yd.
4. 12 cm., 16 cm., 20 cm.

5. Find the lengths of the altitudes drawn from $A$ and
in the triangle shown in Example 2.

6. Find the lengths of the three altitudes of the triangle
Ex. 1.

7. Find, to 2 figures, the lengths of the three altitudes
the triangle of Ex. 3.

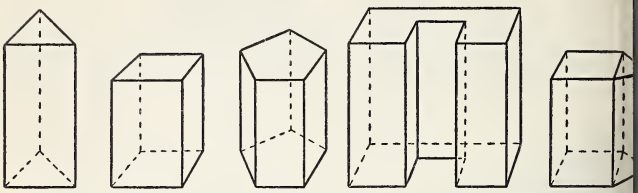
*What geometric forms appear in these pictures?*

Top: Polymer Rubber Corporation, Sarnia, Ontario.
Centre: Western Ontario tobacco field.
Bottom: Great Lakes shipping scene.

(*National Film Board Photos*)

## *Prisms and Pyramids*

Study the solids in the figure below; they are called ***prism***



They look something like cylinders except that their bases a not circular. You see prisms among the apparatus of yo physics laboratory as well as in the forms of buildings, colum and storage bins. Try to discover ways in which the pris shown in the figure resemble one another.

* * *

From the figures, you observe that prisms have

(1) *two identical bases in parallel planes*, and

(2) *rectangular*[1] *sides.*

For the bases of a prism any polygon can be used. Now t to find a formula for the volume of a prism.

* * *

Suppose that the area of the base of the prism in Fig. a 5 square inches. Then, if this prism is filled with water to depth of 1 inch, we have 5 cubic inches of water in the pris A depth of 2 inches would give 10 cubic inches, and so o Thus, if the height of the prism is 8 inches, its volume is cubic inches. In the same way, for a prism whose base is square units and height $h$ units, we get the formula:

[1]Some prisms are oblique, with parallelograms as sides, but we shall study them.

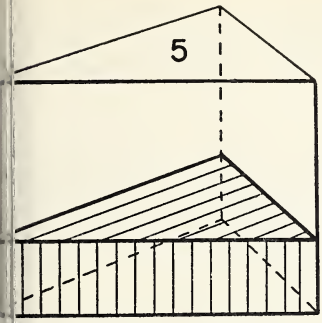
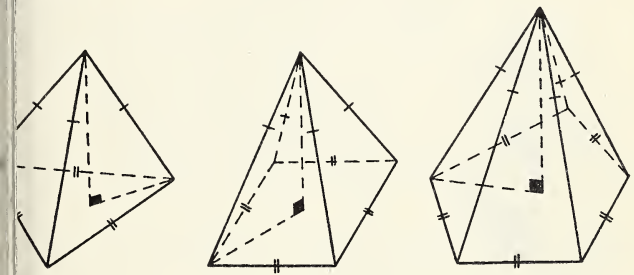



Fig. a

***Volume of Prism***

$$V = bh$$

Next study the solids shown below; these solids are called ***amids.***



h of the pyramids which we shall study has

(1) *a base which is a regular polygon* (that is, a polygon with equal sides and equal angles), and

(2) *sides which are isosceles triangles with a common vertex.*

In order to find the volume of a pyramid, try the following eriment: By means of some thin cardboard and tape, struct a prism and a pyramid with equal heights and bases. is a good idea to make the pyramid first, since the height

of the prism can be adjusted to that of the pyramid qu easily.) Do not attach the base to the pyramid or the t to the prism. In this way, you obtain two hollow, op models.

Now, fill the pyramid with dry sand and empty the conte into the prism. Repeat until the prism is filled with sa How many fillings of the pyramid are needed to fill the pris What do you conclude about the volume of the pyramid?

* * *

If you have performed the experiment carefully, you w find that the prism holds three times as much as the pyram Therefore, if the base of a pyramid is $b$ square units, and t altitude $h$ units, we have the formula:

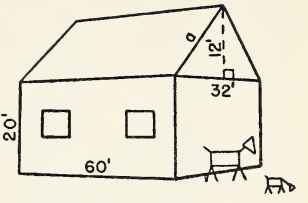
***Volume of a Pyramid***

$$V = \tfrac{1}{3}bh$$

No special formulas are needed for the areas of prisms pyramids. We can readily calculate them by adding areas of the figures which make up the sides and bases.

*Example* 1. A storage barn is constructed as shown in figure. Find its cubical contents. If there are 90 sq. ft. window openings, what area (roof excluded) requires paintir

The barn is in the form of a
m resting on one side. The
s of the prism consist of a
angle surmounted by a tri-
e. If the area of a base is
uare feet, then




$= 32 \times 20 + 16 \times 12 = 832,$
$= bh = 832 \times 60 = 49{,}920.$

Therefore, the volume of the barn is about 50,000 cu. ft. (correct to two figures). If the dimensions were measured correct to the nearest tenth of a foot, we should be justified in reporting the volume as 49,900 cu. ft. (correct to three figures). Do you think it would be reasonable to measure the dimensions of a barn correct to the nearest tenth of a foot? To the nearest hundredth of a foot?

The area to be painted is made up of two sides, and two ends, less the window openings. Therefore, the number of square feet in this area is

$$2\,(20 \times 60) + 2\,(20 \times 32 + 12 \times 16) - 90$$
$$= 2400 + 1664 - 90 = 3974.$$

The area is therefore 3970 square feet, assuming three-figure accuracy, or 4000 square feet assuming two-figure accuracy.

*Question:* If the answer is reported as 3970 sq. ft., correct to 3 figures, by how much may the answer be in error? What is the possible error in the answer 4000 square feet, correct to 2 figures?

*Example* 2. For the barn in Example 1, find the area of the roof.

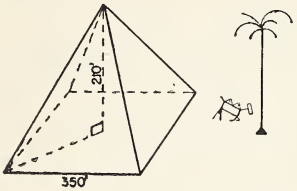
If the width of the roof is $a$ feet, we have from the theorem of Pythagoras,

$$a^2 = 12^2 + 16^2 = 144 + 256 = 400.$$

Therefore, $a = 20$, and the number of square feet in the roof is

$$2\,(60 \times 20) = 2400.$$

*Example* 3. One of the pyramids of Egypt when comple had a square base, with sides 350 feet and a height of 210 f Neglecting inner passages, find the volume of material in pyramid.




We have,

$$V = bh/3,$$
$$= (350^2)\ 210/3,$$
$$= 8{,}575{,}000.$$

Therefore the pyramid c tains 8,580,000 cu. ft. (3-fig accuracy).

## EXERCISES

1. Find the volume of a prism having an altitude of inches and a triangular base whose area is 4.2 sq. in.

2. A pyramid has a square base with sides 6.4 inches. the altitude of the pyramid is 7.5 inches, find its volume corr to 2 figures.

3. Name three objects commonly made in the form o prism.

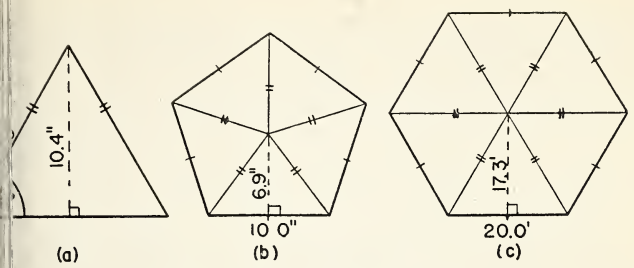
4. Find a building in your community that makes use the pyramidal form. Point out some pyramids in the airv of Ottawa on page 266.

5. Find the volume of a pyramid whose base is 100 sq. and whose altitude is 8.2 cm.

6. Solve the formula $V = bh/3$ (*a*) for $h$, (*b*) for $b$.

7. The base of a pyramid is 80 sq. in. and its volume, cu. in. Find its height.

8. If the volume of a pyramid is 1000 c.c. and the hei 7.5 cm., find the area of its base.



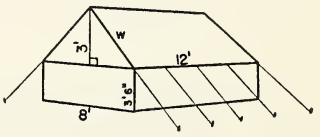

). Fig. (a) shows the base of a prism. If the height of the n is 40.0 in., find its volume.

10. A cement column with height 20.0 feet has bases as vn in Fig. (b). Find (*a*) the volume of cement in the mn, (*b*) the area of the lateral faces.

11. A storage tank is made in the form of a prism with gonal base as shown in Fig. (c). If the tank is 50.0 feet , find its volume.

12. Find the volume of a pyramid with altitude 40 feet base as shown in Fig. (c).

13. For the tent shown in figure, find (*a*) the cubical ents, (*b*) the area of the and ends, (*c*) the width ne side of the roof (w), (*d*) rea of the roof. (Assume t data.)

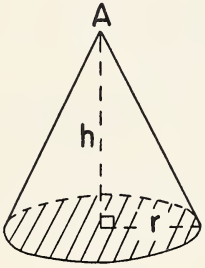



## *Cones and Spheres*

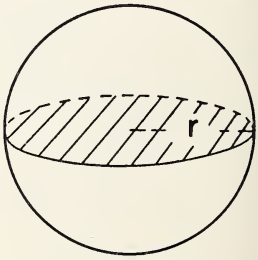
Young Canadians will have no difficulty in recognizing the . Surmounted by a hemisphere of ice cream, cones are and appreciated in every city, town, and village in the

land. If you can manage to picture the cone without the ice cream, you will see that it is very much like a pyramid. Imagine a pyramid whose base is a polygon of many sides all of which are equal. As the number of sides gets larger and the length of each side gets smaller, the base of the pyramid becomes circular. If the number of sides of the base were great enough, you couldn't tell the pyramid from a cone.

The base of a cone is a circle, as the figure shows. The point $A$ is called the ***vertex*** of the cone, and the perpendicular from $A$ to the base is the ***altitude*** of the cone. What is the formula for the volume of the cone?



Cone



Sphere

Since the cone is a special kind of pyramid, we should expect its volume to equal one-third of its base times its height. This is a fact. You can illustrate it by performing the following experiment.

Either obtain a conical paper cup or make a hollow cone from a sector of good, stout paper. Then make a hollow cylinder with an open top and having the same radius and height as those of the cone. Fill the cylinder with sand from the cone. What do you observe? * * *

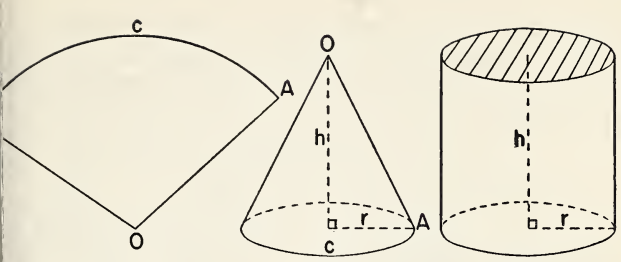



Fig. a Fig. b Fig. c

t takes three cones of sand to fill the cylinder.

***Volume of a Cone***

$$V = \tfrac{1}{3}\pi r^2 h$$

he area of the lateral surface of a cone can be readily d. You notice from Figs. a and b that, if the cone is split g $OA$ and opened out, its surface becomes a sector of a e with radius $OA$. Moreover, the arc of this sector is l to the circumference of the base of the cone. Denoting lant height of the cone by $s$ units, the radius of its base by its, and the area of the lateral surface by $A$ units, we have

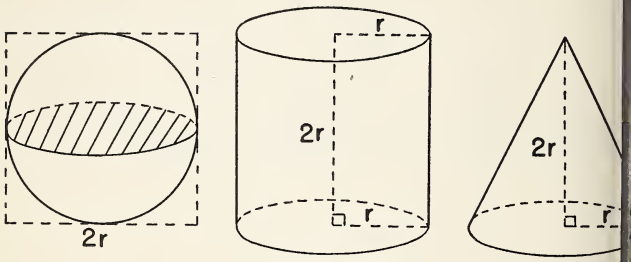
$$A = \tfrac{1}{2} \cdot \text{arc} \cdot \text{radius} = \tfrac{1}{2} \cdot 2\pi r \cdot s = \pi rs.$$

***Area of Lateral Surface of Cone***

$$A = \pi r s$$

he sphere, commonly called a ball, comes into everyday n many ways. Here is a method of finding the formula ts volume. Find a ball about three inches in diameter get a tin can into which the ball just fits. Make the

height of the can equal to the diameter of the ball. Then m
a cone of waxed paper with the same height and radius as
cylinder.[2] The sphere, cylinder, and cone should appea
in the figure.




Place the sphere in the cylinder and add one cone of wa
What do you observe? * * * You will find
the sphere and the cone of water just fill the cylinder. Th
fore, we have

volume of cylinder = volume of cone + volume of spher

Hence, if the volume of the sphere is $V$ units, we obtain

$$\pi r^2 (2r) = \tfrac{1}{3} \pi r^2 (2r) + V,$$

or

$$V = 2 \pi r^3 - \tfrac{2}{3} \pi r^3 = \tfrac{4}{3} \pi r^3.$$

***Volume of Sphere***

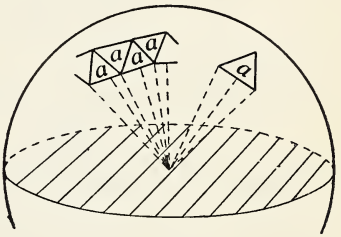
$$V = \tfrac{4}{3} \pi r^3$$

NOTE: In more advanced mathematics, we have much more rig
methods for obtaining formulas than those used in this chapter. Our pr
experimental methods are intended merely to make the formulas seem re
able. You are asked to accept on faith the fact that the formulas c
proved with complete rigour.

[2]Such sets can be purchased from laboratory supply firms.

A suggestive method of finding the formula for the area of phere is the following: Imagine the sphere to be made up small pyramids whose equal bases completely cover the face of the sphere and whose vertices are at its centre as wn in the figure. The altitude of each pyramid is very e in value to $r$, the radius of the sphere.

Now if the base of each amid is small enough, we think of it as being flat. reover these tiny, needle- pyramids fill the sphere hat the volume of the re equals the sum of the mes of the pyramids.




f each little base is $a$ units in area, then the volume of each pyramid is $ar/3$ units. This gives the number of units in volume of the sphere $= ar/3 + ar/3 + ar/3 + \cdots$.

$$= \frac{r}{3}(a + a + a + \cdots),$$

$$= \frac{r}{3}(\text{number of units in area of sphere}).$$

Therefore, if the area of the sphere is $A$ units, we have

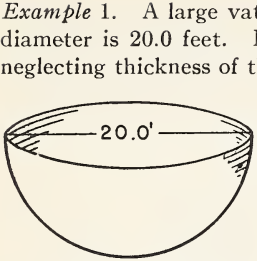
$$V = \frac{r}{3}A,$$

$$A = \frac{3}{r} \cdot V = \frac{3}{r}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2.$$

***Area of Sphere***

$$A = 4\pi r^2$$

*Example* 1. A large vat is in the form of a hemisphere wh diameter is 20.0 feet. Find the volume and area of the v neglecting thickness of the material.




We have

$$V = 2\pi r^3/3 = 2(3.14)\ (1000)/3,$$
$$= 2093\tfrac{1}{3}.$$

Therefore, the volume is 2090 ft., correct to 3 figures.

The area is given by

$$A = 2\ \pi\ r^2,$$
$$= 2\ (3.14)\ (100) = 628.$$

Hence the area is 628 sq. ft. to 3 figures.

*Example* 2. It is desired to make a conical funnel with capacity of 1250 c.c. and the diameter of its base 14.0 c Find the altitude to 3 figures.

We have $V = \pi\ r^2\ h/3,$

whence $h = 3\ V/\pi\ r^2,$

$$= 3750/(22/7)49,$$
$$= 3750/154 = 24.4, \text{ to 3 figures.}$$

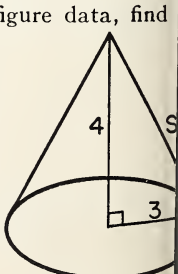
Therefore, the funnel is 24.4 cm. high.

*Example* 3. The altitude of a conical tank is 4 feet and radius of its base, 3 feet. Assuming 3-figure data, find area of the lateral surface of the tank.

From the theorem of Pythagoras, we have

$$s^2 = 3^2 + 4^2 = 25,$$

whence $s = 5$. This gives




$$A = \pi rs = \pi \cdot 3 \cdot 5 = 15 \times \frac{22}{7} = 47\tfrac{1}{7}.$$

Therefore the required surface is 47.1 sq. ft., to 3 figure

## EXERCISES

(*In the following exercises, take* $\pi = 22/7$.)

1. If the radius of a sphere is 7 inches, find (*a*) its volume, its area. (Assume exact data.)

2. If the altitude of a cone is 30 cm. and the diameter of its se, 14 cm., find its volume. (Assume exact data.)

3. Solve the formula $V = \pi r^2 h/3$ (*a*) for $r$, (*b*) for $h$.

4. If the radius of a sphere is doubled, what change is duced in (*a*) its volume, (*b*) its area?

5. The radius of the moon is about $\frac{1}{4}$ that of the earth. w does the volume of the earth compare with that of the on? How do their areas compare?

6. A conical pile of coal is 12 feet high and the diameter of base is 16 feet. Find, to 2 figures, the volume of the pile.

7. Assuming the earth to be a perfect sphere with radius 0 miles, find, to 2 figures, (*a*) its area, (*b*) its volume.

8. Find, to 3 figures, the *total* surface of the cone of Example 3.

9. What is the volume of the largest sphere that can be ced inside a cube whose edges are 14 inches?

10. The height of a conical paper weight is 3.6 cm. and the meter of its base 4.2 cm. Find, to 2 figures, the number of h paper weights that could be cast from 1000 c.c. of metal.

11. Find the lateral surface of a cone having radius 12 ts and slant height 14 units. (Assume 3-figure data.)

12. What will it cost to gild a ball with radius 7.70 cm. if rate is 2 cents per square centimetre? (3-figure accuracy)

13. Find, to 3 figures, the number of cubic feet in a spherical oline storage tank whose diameter is 21.0 feet. If 1 cubic t equals 6.23 gallons, find the capacity of the tank in gallons.

14. A conical float with a closed top has an altitude of feet and a diameter of 10 feet. Find, to 3 figures, its to surface.

## REVIEW EXERCISES

### *A*

1. Mr. Acres measures the dimensions of a rectangu field, finding it to be 40 rods long and 32 rods wide. If th measurements are correct to the nearest rod, how should M Acres report the area of the field?

2. In the measurement of Ex. 1, what is the possible er in the length and width as obtained by Mr. Acres? What the possible error in the area of the field, reported to t figures?

3. George measures the inside dimensions of a chalk b obtaining the following: length 15.5 cm., width 9.8 cm., dep 8.7 cm. What is the volume of the box? By how much mig your answer be in error?

4. In order to dig a well, Mr. Drye excavates a cylindri hole $24\frac{3}{4}$ feet deep and $3\frac{1}{2}$ feet in diameter. Find, to nearest cubic foot, the volume of earth removed. ($\pi = 22$

5. A closed tin can is 36.5 cm. high and 28.0 cm. in diame Neglecting any over-lapping, find the area of the tin requi to make the can. ($\pi = 22/7$)

6. A cement column has the form of a prism 20 ft. h with an octagonal base. If the area of the base is 3.6 sq. find the volume of the column.

7. A metal weight is cast in the form of a pyramid wit square base. If the altitude of the pyramid is 10.0 cm. the side of the base is 15.0 cm., find the volume of the casti

8. A conical funnel is 32.6 cm. high with a base diame of 21.7 cm. Find the volume of the funnel. (Take $\pi = 22$

9. A spherical rubber balloon is blown up until its diameter 36.6 inches. Find, to 3 figures, the volume of air in the lloon.

10. Find the area of a triangular piece of tin with sides (*a*) in., 17 in., and 21 in.; (*b*) 10 in., 24 in., and 26 in.

11. Find the lengths of the three altitudes of the triangle Ex. 10(*b*).

12. A sector is cut from a circular metal sheet having dius 1.40 ft. Find the area of the sector (*a*) if its angle is 4°, (*b*) if its arc is 1.20 feet long.

13. The slant height of a wooden cone is 3.00 feet and the dius of its base is 2.10 feet. Find, to 3 figures, the total ea of the surface of the cone.

*B*

14. Find, to 2 figures, the volume of the trough shown in g. a.

15. Find the volume of the piece of tile shown in Fig. b.

16. A pail is made by removing a small cone from a larger llow cone and then adding a bottom to the remainder ig. c). Find, to 2 figures, the volume of the pail.

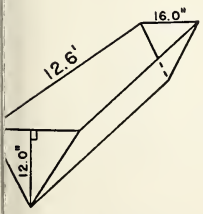



Fig.a

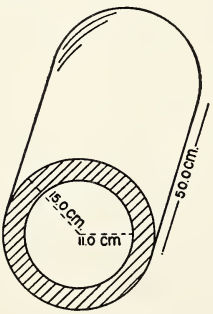



Fig.b

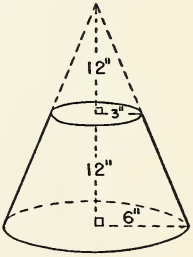



Fig.c

**17.** The dome of a building is in the form of a hemisphe with diameter 20.0 feet. Find, to the nearest dollar, the co of painting the dome at 5 cents per square foot.

**18.** In making a certain type of washer, a hole one inch diameter is punched from the centre of a metal disc two inch in diameter. What fraction of disc is punched out?

**19.** Farmer Oates wishes to construct a cylindrical silo hold 60 tons of ensilage. If the diameter of the base is to 14 feet and one ton of ensilage occupies 50 cu. ft., find t approximate height of the silo.

**20.** A bakeshop offers two sizes of pies for sale. Sm pies 6 inches in diameter cost 10 cents, and larger pies inches in diameter cost 30 cents. If both kinds of pie are t same thickness, how does the volume of a large pie compa with that of a small one? Would it be economical to b the large pies at 45 cents?

**21.** A circular fish pond 9.2 feet in diameter is surround by a tiled path 1.8 feet wide. If it costs 70 cents per sq. to lay the tile, find, to the nearest dollar, the cost of the pat

**22.** The storage barn shown in Fig. d has a volume of 31,8 cu. ft. Find the length of the barn, $l$ ft. What is the area its sides and ends?

**23.** For the barn in Fig. d, find the area of the roof.

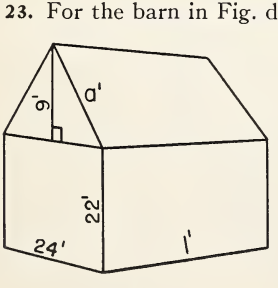



Fig. d

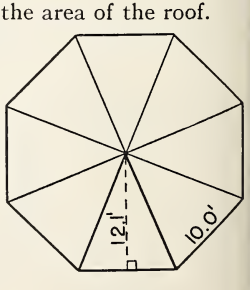



Fig.e

**24.** A reservoir is to be made with an octagonal base as own in Fig. e and a volume of 15,000 cu. ft. Find its height the nearest foot.

**25.** Find the volume of a pyramid with base as shown in . e and height 60 ft.

**26.** A buoy consists of a cylinder with two cones attached shown in Fig. f. Find (*a*) the volume of the buoy, (*b*) the al area.

**27.** Fig. g shows a buoy made in the form of a cylinder h two hemispheres added. Find its volume and area.

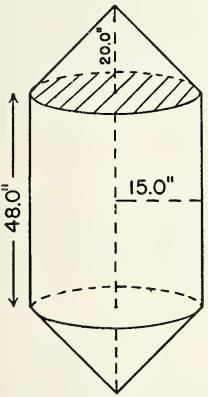



Fig.f

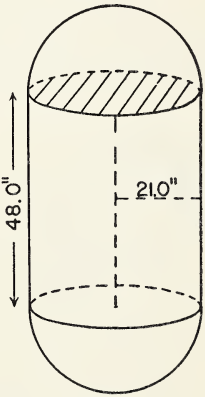



Fig.g

**28.** What is the effect on the area of a sphere when its ius is (*a*) tripled? (*b*) halved? (*c*) decreased to $\frac{3}{4}$ of its ginal value?

**29.** Show that the volume of an orange with diameter 3 hes is $9\pi/2$ cu. in. If the skin of this orange is $\frac{1}{4}$ in. thick,

find the edible volume. What per cent of the orange is edib
If the skin of the orange is $\frac{3}{8}$ in. thick, what per cent of t
orange is edible?

**30.** A spherical storage tank has a diameter of 25.0 fe
If one cubic foot contains 6.23 gallons, find the volume of t
tank in gallons.

**31.** Find, to the nearest dollar, the cost of painting the ta
of Ex. 30 at 5 cents per sq. ft.

**32.** It is required to make a conical vat with a volume
400 cu. ft. and a top 12.0 feet in diameter. Find its height.

**33.** A contractor wishes to estimate the number of cu
yards in a conical pile of gravel. He judges the height of
pile to be 24 feet, and by pacing finds the circumference of
base of the pile to be about 40 yards. What would be a reas
able estimate of the number of cubic yards in the pile?

**34.** What is the effect on the volume of a sphere when
radius is (*a*) doubled? (*b*) halved? (*c*) tripled? (*d*)
creased to 3/2 times its original length?

**35.** What is the effect on the volume of a tin can when
its radius is doubled? (*b*) its height is doubled? (*c*) its heigh
doubled and its radius is halved? (*d*) its height is doubled a
its radius reduced to $\frac{3}{4}$ of its original value?

**36.** The crater of a volcano is approximately conical
form. If the depth is about 1500 feet and the distance arou
the rim about 1760 yards, find, to 2 figures, the volume of
cavity in cubic yards.

CHAPTER XV

# MORE EVERYDAY USES OF ARITHMETIC

*"The time has come," the Walrus said,*
*"To talk of many things."*[1]

Lewis Carroll

Mathematics has many uses. Among them is the use we ke of it in our everyday affairs. In Book I, you had some ıctice in the arithmetic used by a merchant as well as in the culation of simple interest, fire insurance, and property es. In this chapter, we discuss four other topics which ect us all, as citizens. In fact, you will, in a very few years, translating their theory into practice. The mathematics olved is simple arithmetic, with two or three of the results ıdensed into algebraic formulas.

## *Shall I Buy on the Installment Plan?*

When Mr. Davidson decided to buy a car he found that the del he wanted could be bought for a cash payment of $1065. was also told that he could buy the car on the install- nt plan by paying $450 cash and $45 a month for 15 months. . Davidson then went home to discuss with his family ether he should buy on the installment plan or wait until could save enough money to pay for the car in cash.

When expensive articles such as cars, furniture, radios, and rigerators are bought, the purchaser may find that it is

[1] From *Through the Looking-Glass*, published by Macmillan & Company ited, London, England.

impossible or inconvenient for him to pay the full cost of th article in cash. Then he is usually given the privilege paying in installments. On account of this privilege, th installment cost is higher than the cash price. Is the advanta of having the article in advance worth the extra cost? Th is the question which calls for the exercise of judgment on th part of the buyer and it is worth while for him to do a litt arithmetic in order to reach a decision.

*Example* 1. An electric stove can be bought for $200 ca or for $100 now and $110 a year from now. Consider th second offer. On it the dealer gets $100 instead of the fu price of $200 now. He is therefore making a loan of $100 to th customer for one year. At the end of the year he gets back th loan with $10 interest. On $100 for one year the customer wou pay $10 interest, or he would pay interest at 10% per annu

*Example* 2. The dealer offers to sell the same stove on t installment plan for $100 down, $54 at the end of 6 months a $54 at the end of a year. On this plan the payments a reckoned as follows:

| | |
|---|---|
| Cash | $100 |
| First installment | $ 54 |
| Second installment | $ 54 |
| Installment price | $208 |
| Cash price | $200 |
| Installment charge | $ 8 |

What rate of interest does this installment charge rep sent? Since the cash value of the stove is $200 and the dea receives $100 as a down payment, he is really lending $100 the purchaser for 6 months. At the end of 6 months the dea gets another payment of $54; his loan to the customer for t remaining 6 months is therefore $46. It follows that t purchaser has the use of $146 for 6 months and for this pays $8 interest. The rate of interest is then found from t following simple interest formula.

$$i = prt$$

We have $8 = 146 \times r \times \frac{1}{2}$.

Hence $r = \frac{8 \times 2}{146} = 0.11$ nearly.

The rate is therefore 11% per annum, very nearly.

*xample* 3. Miss Cameron wished to buy a radio which she w advertised at $10 down and $5 a month for 12 months. ıe cash price of the radio was $63. She was therefore faced th the problem of deciding whether to buy it on the install- ent plan or wait a few months until she could save the $63.

Let us first calculate the installment charge, which is the terest she would pay to the dealer if she bought by install- ents.

| | |
|---|---|
| Cash payment | $10 |
| 12 payments of $5 each | 60 |
| Installment price | 70 |
| Cash price | 63 |
| Installment charge | $ 7 |

ıe rate of interest represented by this charge can be found proximately by extending the method of the two preceding amples. Since the cash value of the radio is $63, then if iss Cameron pays $10 the company lends her $53 for the st month. If, at the beginning of the second month she ys an installment of $5, her loan for this month is $48. For e third month her loan would be $43, and so on. Her total ın is the same as a loan ***for one month*** of

53 + 48 + 43 + 38 + 33 + 28 + 23 + 18 + 13 + 8 + 3)

$308. On this loan her interest would be the $7 installment

charge. The rate of interest is found from the formula $i = pr$ or

$$7 = 308 \times r \times 1/12,$$

which gives $r = 3/11 = 0.27$ to 2 decimal places. This mean that, if she bought the radio on the installment plan, she woul pay interest at about 27% per annum.

From these examples and others which you may lear about, you can estimate the merits and demerits of instal ment buying. It is sometimes a convenience for a purchas to divide his payments into installments. He may, f example, need a car in his business before he is able to pay t cash price for it. It will be found, however, that he usual pays a high rate of interest for the loan consisting of t unpaid installments. For this reason it is advisable, whe possible, for a purchaser to anticipate his needs and to sa regular amounts from his income so as to pay cash for h purchases. While his savings are accumulating in a bank, is then *receiving* a modest rate of interest instead of *paying* high rate after the article is bought.

## EXERCISES

Find the installment charge in each of the following thr offers.

1. A fur coat priced at \$185 is offered for \$75 down a \$10 a month for 12 months.

2. A bicycle priced at \$39.50 is offered for \$5 down a \$2 a week for 20 weeks.

3. A radio priced at \$95 is offered for \$25 down and \$6 month for 14 months.

4. A refrigerator priced at \$220 cash is bought for 3 p ments of \$78, one now, one at the end of 6 months, and at the end of a year. Find the rate of interest which buyer pays on the unpaid balance of the cost.

5. Find the rate of interest which the installment charge presents in Ex. 1.

6. A canoe priced at $116 can be bought on the installment an for $15 down and $18 a month for 6 months. Find (*a*) e installment charge, (*b*) the rate of interest paid.

7. A $418 motorcycle is bought on the installment plan r $55 down and $33 a month for 12 months. Find (*a*) the stallment charge, (*b*) the rate of interest paid.

8. In selling a used car, a dealer offers a choice of a cash ice of $510 or an installment price of $175 down and $30 a nth for 12 months. On the installment plan find (*a*) the stallment charge, (*b*) the rate of interest.

9. In the example at the beginning of this section, find the te of interest which Mr. Davidson would pay if he bought e car on the installment plan.

## *Unpleasant but Necessary—the Income Tax*

The tax on incomes is one of the most important sources the large amounts of revenue which are needed by the leral government. The rates and exemptions vary according the national need; during the Second Great War people of derate income paid from $\frac{1}{5}$ to $\frac{1}{4}$ of their total income in xes. In all cases, however, a certain amount of a taxpayer's come, depending on the number of his dependents, is exempt m taxation. On his taxable income he pays at a rate ich increases with the size of his income, so that very ge incomes are heavily taxed, even to the extent of half or ree-quarters of the income. The rates given in the following ample are those which applied during some of the war years.

*ample.* Mr. J. Citizen, who is married and has 3 dependent ildren, found, after several evenings' computation, that his xable income was $2885. He then attacked the problem

of finding his tax which, he learned, was made up of two part a normal tax and a graduated tax. His income tax for instructed him to compute his normal tax at 7% of his taxab income and then to deduct $28 for each dependent chil His graduated tax was based on his taxable income less $66 according to the following schedule:

On the first $500 of taxable income the rate of tax is 30%

between $500 and $1000 the tax is $150 + 33% of th excess over $500;

between $1000 and $2000 the tax is $315 + 37% of th excess over $1000;

between $2000 and $3500 the tax is $685 + 41% of th excess over $2000;

between $3500 and $5000 the tax is $1300 + 45% of th excess over $3500.

From his graduated tax Mr. Citizen was instructed to dedu tax credits of $150 on account of married status and $80 f each dependent child.

When Mr. Citizen finally got his tax computed, his arit metic looked like this.

| | | |
|---|---|---|
| Taxable income . . . . . | | $2885.00 |
| Normal tax $2885 × 0.07 . . | $201.95 | |
| Tax credit for 3 dependent children $28 × 3 . . . . | $ 84.00 | |
| | | $ 117.95 |
| Graduated tax on $(2885 − 660) or $2225 is $(685 + 225 × 0.41) = $(685 + 92.25) . . . | $777.25 | |
| Tax credits: married status $150 3 dependents $240 | $390.00 | |
| | | $ 387.25 |
| Total normal and graduated tax . . . | | $ 505.20 |

The size of this tax gave Mr. Citizen a start until he realized at, in accordance with the government's plan of *deductions at e source*, most of it had already been deducted from his lary before he received it.

On some of the later income tax forms, tables are provided hich give the answers to many of the calculations so that the xpayer has less arithmetic to do in filling out his form.

### EXERCISES

**1.** Using the rates quoted in the foregoing example find the come tax paid by Mr. Hardprest, who has a family of 5 de-ndent children and a taxable income of $2432.

**2.** Using the same rates, find the tax paid by Mr. Affluent, ho has one child and a taxable income of $4569.

## 4 *Matter of Interest—Simple and Compound*

In order to pay for repairs on his house, Mr. Fleming orrowed $300 from The Burdenless Loan Company, agreeing pay interest at 5% per annum. His interest bill for one ear then amounted to 5% of $300 or $15. Instead of paying is interest, Mr. Fleming allowed the loan to run for another ear. In the second year, the principal of his debt was $315 nd the second year's interest was 5% of $315 or $15.75. At e end of the second year, he discharged the whole debt by payment of $330.75. This ***amount*** included his original ***rincipal*** of $300 and two years' interest of $30.75.

Many business transactions consist of the borrowing and nding of money, which often remains invested over a period years. It is then the practice to add the accumulated terest to the principal at the ends of stated intervals of me. The interest may, for example, be added yearly or alf-yearly; the ***interest period*** is then said to be a year or a

half year, respectively. Whenever interest is added to princi pal, the resulting amount becomes the principal for the nex interest period. Interest calculated in this way is calle ***compound interest.*** This name distinguishes it fro ***simple interest,*** in which the principal remains fixed throug out the duration of the loan.

The use of simple interest is confined to short ter loans, usually of a year or less. For loans of longer duratio such as those made to municipalities and other corporation compound interest is the general rule. The following exampl show how compound interest is calculated.

*Example* 1. Find the interest on \$400 for 4 years at 5% per annum compounded annually.

The first year's interest = \$(400 × 0.05) = \$ 20.
The second year's principal = \$(400 + 20) = \$420.
The second year's interest = \$(420 × 0.05) = \$ 21.
The third year's principal = \$(420 + 21) = \$441.
The third year's interest = \$(441 × 0.05) = \$ 22.0
The fourth year's principal = \$(441 + 22.05) = \$463.0
The fourth year's interest = \$(463.05 × 0.05) = \$ 23.1
The amount at the end of the fourth year
= \$(463.05 + 23.15) = \$486.20.
Hence the compound interest for 4 years
= \$(486.20 − 400) = \$86.20.

*Example* 2. Find the amount of \$175 for $1\frac{1}{2}$ years at 3% pe annum, compounded semi-annually.

In this problem the interest period is a half year. Th rate of 3% per annum is equivalent to $1\frac{1}{2}$% per half year.

The first half year's interest
= \$(175 × 0.015) = \$2.625.
The second half year's principal
= \$(175 + 2.625) = \$177.62

he second half year's interest

$$= \$(177.625 \times 0.015) = \$2.664.$$

he third half year's principal

$$= \$(177.625 + 2.664) = \$180.289.$$

he third half year's interest

$$= \$(180.289 \times 0.015) = \$2.704.$$

he amount at the end of $1\frac{1}{2}$ years

$$= \$(180.289 + 2.704) = \$182.99.$$

**EXERCISES**

Find the compound interest on:

1. \$240 for 2 years at 4% per annum, compounded yearly;
2. \$300 for 3 years at $2\frac{1}{2}$% per annum, compounded yearly;
3. \$80 for $1\frac{1}{2}$ years at 5% per annum, compounded half-arly;
4. \$220 for 2 years at 4% per annum, compounded half-arly;
5. \$75 for 2 years at 3% per annum, compounded half-arly;
6. \$100 for 1 year at 4% per annum, compounded quarterly;
7. \$1000 for $2\frac{1}{2}$ years at 4% per annum, compounded arly (Compute the last half year's interest at 2%);
8. \$125 for $1\frac{1}{2}$ years at 5% compounded yearly.

## *The Algebra of Simple Interest Reviewed*

If a sum of \$100 bears simple interest for 2 years at 3% per num, the interest is $\$(100 \times 0.03 \times 2)$ or \$6 and the amount

$$\$100 + \$(100 \times 0.03 \times 2) \text{ or } \$100(1 + 0.03 \times 2)$$

hich gives \$106. This example reminds you of the formula

in which this reasoning was crystallized. First let us refres our memories regarding the rate. A rate of 3% means $ on a hundred or 3¢ on a dollar. In other words, a rate of 3% means 0.03 per unit. Conversely a rate of $r$ per unit mean $100r\%$. In our formulas we shall employ $r$ as the rate pe unit.

If, now, a principal sum of $p$ dollars bears simple interes for $t$ years at the rate $r$ per unit, the interest (see Book p. 332) is

$$i = p\,r\,t$$

and the amount is $A = p + p\,r\,t = p\,(1 + rt)$.

## *The Algebra of Compound Interest*

Suppose that a sum of $\$p$ is invested at the rate of $r$ p unit and that the interest is compounded annually for $n$ year The amount at the end of 1 year is the amount at simp interest, that is $\$p(1 + r)$, and this becomes the principal f the second year. Now take a second look at this formul there is magic in the letters! It says that the amount, at ra $r$ for 1 year, of any sum $\$p$ is obtained by multiplying $p$ $(1 + r)$. So the amount of \$600 for 1 year is \$600 $(1 +$ and the amount of $\$3q$ for 1 year is $\$3q\,(1 + r)$. Now our problem, in which the original principal was $\$p$, t principal during the second year is $p\,(1 + r)$ dollars; hen the amount at the end of the second year is $p\,(1 + r)\,(1 +$ dollars. Since this becomes the third year's principal, t amount at the end of 3 years is

$$p\,(1 + r)\,(1 + r)\,(1 + r)$$

dollars. Let us write the results so that they can be easi compared:

Amount at the end of 1 year $= \$p\,(1 + r)$;
Amount at the end of 2 years $= \$p\,(1 + r)\,(1 + r)$;
Amount at the end of 3 years $= \$p\,(1 + r)\,(1 + r)\,(1 +$

hat would be the amount at the end of 4 years? at the end 15 years? at the end of $n$ years? We see that, in each swer, the factor $1 + r$ is raised to the power indicated by e number of years. Thus, the amount at the end of 27 ars is $\$p\,(1 + r)^{27}$ and the amount at the end of $n$ years is $(1 + r)^n$.

***The foregoing reasoning holds equally well if the terest period is different from a year provided that r is te for one interest period.*** Hence we can now condense e result into a formula which gives the amount of a sum of $p$ llars for $n$ interest periods at the rate $r$ per unit for each erest period.

***Amount at Compound Interest***

$$A = p(1 + r)^n$$

*ample.* Find the interest on \$120 for 3 years at 4% per num compounded half-yearly.

The rate of 4% per annum is equivalent to 2% per half ar. Since the interest is compounded half-yearly, there are nterest periods in 3 years. Hence we apply the foregoing mula with $p = 120, r = 0.02$, and $n = 6$. This gives

$$A = 120\ (1.02)^6.$$

remains to carry out the multiplication. We shall have nething to say later about such computations. For the sent, you had better convince yourself that you can do the ltiplication in a reasonably short time and get the correct wer. Remember that a computor is judged not merely whether he knows the formulas but also by whether he gets correct answers by using the formulas. We find that in product of $(1.02)^6$ the first 6 digits are sufficient for our pose and that

$$A = 120 \times 1.12616 \text{ or } 135.14.$$

The amount is therefore \$135.14 and the compound interest \$15.14.

## EXERCISES

### *A*

(*These may be answered orally*)

Without working out the answer, express as a formula t amount of each of the following:

**1.** \$50 for 10 years at 5% per annum, compounded year

**2.** \$100 for 15 years at 3% per annum, compounded year

**3.** \$1200 for 5 years at $3\frac{1}{2}$% per annum, compounded year

**4.** \$260 for 8 years at 4% per annum, compounded h yearly;

**5.** \$130 for 7 years at 3% per annum, compounded h yearly;

**6.** \$750 for $8\frac{1}{2}$ years at 3% per annum, compounded h yearly;

**7.** \$3600 for $10\frac{1}{2}$ years at 5% per annum, compounded h yearly;

**8.** \$165 for $4\frac{1}{2}$ years at $2\frac{1}{2}$% per annum, compounded h yearly.

### *B*

**9.** Find the amount of \$100 for 3 years at 5% per ann compounded yearly.

**10.** Find the amount of \$250 for 4 years at 4% per ann compounded yearly.

**11.** Find the amount of \$600 for $1\frac{1}{2}$ years at 3% per annum, mpounded half-yearly.

**12.** Find the compound interest on \$2500 for 2 years at 4% r annum, compounded half-yearly.

**13.** Find the compound interest on \$800 for 3 years at $4\frac{1}{2}$% r annum, compounded yearly.

**14.** Find the compound interest on \$90 for 5 years at 2% r annum, compounded half-yearly.

**15.** Construct a formula for the interest, compounded arly, on $p$ dollars for $n$ years at the yearly rate of $r$ per it.

**16.** Construct a formula for the amount of $2p$ dollars for $n$ ars at 5% per annum, compounded half-yearly.

**17.** Construct a formula for the amount of \$100 for 12 years the yearly rate of $r$ per unit, compounded half-yearly.

**18.** Make estimates of the compound interest on \$100 (*a*) 10 years at 5% per annum, compounded yearly, (*b*) for 10 ars at 5% per annum, compounded half-yearly, (*c*) for 20 ars at 2% per annum, compounded yearly, (*d*) for 20 yeais 2% per annum, compounded half-yearly. Keep your imates for comparison with the answers which you will er learn to read from tables.

## *Present Value*

Read the two following problems carefully.

**1.** Mr. Pickwick wishes to provide for an expenditure of 000 to redecorate his clubhouse 5 years from now. How ich money does he need to invest now at compound interest 3% per annum, in order that it may amount to \$1000 in 5 ars?

2. Mr. Micawber confidently expects a legacy of $1000 "turn up" after 5 years. He wishes to use the legacy no towards the purchase of a house. If money can be invest at 3% per annum, for what sum should he sell the legacy?

Is there any difference in the mathematical solutions these two problems? * * * You see that th are really the same problem. In each one the question m be phrased: What is the cash value or the value ***now*** of $10 paid 5 years from now? This question of ***present value*** is much importance to bankers and others who have to ma financial calculations.

After you have tried to solve the problem yourself, co pare your solution with the following:

Suppose the required present value is $x$ dollars. Since in 5 years at 3% per annum compounded yearly amounts $\$(1.03)^5$, then $x$ dollars will amount to $x$ times as much, th is to $x\,(1.03)^5$ dollars. Therefore we should have

$$x\,(1.05)^5 = 1000,$$

and

$$x = 1000/(1.03)^5.$$

Now this result requires a bit of multiplication and divisi With a little concentration and perseverance we find t answer to be $862.61.

Now it is time to call on the letters to arrange themsel into a formula which will give the present value. The fo going example provides the method. Denote the rate unit by $r$, the number of interest periods by $n$, the amount $A$, and the present value by $P$. Since, in $n$ periods at rate the sum $P$ will amount to $P\,(1 + r)^n$, we have

$$P\,(1 + r)^n = A.$$

This gives the following formula for $P$.

***Present Value***

*(assuming compound interest)*

$$P = \frac{A}{(1 + r)^n}$$

The difference between the amount and the present value is metimes called the ***true discount.*** It is the interest on the esent value, which makes the latter accumulate to the quired amount.

*ample.* Mr. Simpson owes a debt of \$250, which becomes e in 2 years. If interest is reckoned at 3% per annum, mpounded half-yearly, find the present value of this amount d the discount which Mr. Simpson should be allowed if he ys the debt now.

In this problem, $A = 250$, $r$ (the rate per half year) = 0.015, and $n$ (the number of half years) = 4. Hence

$$P = \frac{250}{1.015^4}.$$

On doing the arithmetic, we find $P = \frac{250}{1.06136} = 235.55$.

Hence Mr. Simpson could discharge the debt now by paying 35.55. The true discount which he should be allowed is \$250 \$235.55 or \$14.45.

## EXERCISES

### *A*

*(These may be answered orally)*

Without working out the answer, express as a formula the esent value of:

1. \$200 due in 2 years, the interest rate being 3% p annum, compounded yearly;

2. \$350 due in 8 years, the rate being 4% per annu compounded yearly;

3. \$185 due in 10 years, the rate being $3\frac{1}{2}$% per annu compounded yearly;

4. \$820 due in 12 years, the rate being 4% per annu compounded half-yearly;

5. \$1000 due in 15 years, the rate being 3% per annu compounded half-yearly;

6. \$425 due in 7 years, the rate being $2\frac{1}{2}$% per annu compounded half-yearly;

7. \$105 due in $6\frac{1}{2}$ years, the rate being 5% per annu compounded half-yearly;

8. \$640 due in $4\frac{1}{2}$ years, the rate being $3\frac{1}{2}$% per annu compounded half-yearly.

*B*

9. What is the present value of \$105 due 1 year from no reckoning interest at 5% per annum?

10. Find the present value of \$200 due 3 years from now 2% per annum, compounded yearly.

11. What sum deposited now at 4% per annum will amou to \$1000 in 3 years?

12. Find the sum which, at 5% per annum compound half-yearly, will amount to \$1000 in $2\frac{1}{2}$ years.

13. A man owes a debt of \$250 which will become due in years. He is given the privilege of paying the debt no Find the true discount which he should be allowed if mon is worth 3% per annum compounded yearly.

**14.** Construct a formula for the present value of $D$ dollars e $q$ years from now at 4% per annum, compounded half-arly.

**15.** Without making the calculations, estimate the present lue of (*a*) \$100 due in 2 years at 4% per annum, compounded arly, (*b*) \$100 due in 2 years at 4% per annum, compounded lf-yearly, (*c*) \$1000 due in 10 years at 3% per annum, com-unded yearly, (*d*) \$10,000 due in 20 years at 4% per annum, mpounded yearly. Keep your estimates for comparison th results which you will later learn to read from tables.

## *Our Slaves Do the Drudgery—Compound Interest Tables*

The practice which you have now had in computing com-und interest, amount, and present value have shown you at much work is involved in these computations. Moreover, e number of interest periods in each of your problems was all. In the problems which arise when towns, cities, or dustrial corporations borrow money, the number of years volved is often 5, 10, 20, or even more. In order to relieve e clerks from the long calculations that would otherwise be cessary, tables are constructed which give the amounts and esent values. A short table of this kind is given on pages 6–407.

*ample* 1. Find, by using the tables, the amount of \$500 · 20 years at 4% per annum, compounded yearly.

From the table we read that $1.04^{20} = 2.19112$. This means that a sum of 1 dollar will amount in 20 years at 4% to 2.19112 dollars. Hence \$500 will amount to 500 times as much, that is to \$1095.56.

*ample* 2. Find, by using the tables, the present value of 00 due 8 years from now, if interest is calculated at 4% per num, compounded half-yearly.

There are 16 interest periods and the rate is 2% per perio
From the table we read that $1/1.02^{16} = 0.728446$. Th
means that the present value of \$1 for 16 periods at ra
2% is \$0.72846. Hence the present value of \$800 is 8
times as much, or \$582.76.

### EXERCISES

( *Use the tables*)

1. Find the amount of \$750 invested for 15 years at 3
per annum, compounded yearly.

2. Find the amount of \$1225 for 27 years at 3% per annu
compounded half-yearly.

3. Find the amount of \$3760 for 18 years at 4% per annu
compounded half-yearly.

4. Find the present value of \$1575 due in 24 years at $2\frac{1}{2}$
per annum, compounded yearly.

5. Find the present value of \$2800 due in 14 years at 4
per annum, compounded half-yearly.

6. Find the present value of \$13,000 due in 16 years
$1\frac{1}{4}$% per half year, compounded half-yearly.

7. Use the table to find the answers to Ex. 18, page 30
and compare these with the estimates which you made.

8. Use the table to find the answers to Ex. 15, page 31
and compare these with the estimates which you made.

## *Life Insurance—What Is It and Why Buy It?*

It is an ingrained characteristic of human beings to b
concerned for their future safety and security. One of t
foremost considerations in our planning for such security

*surance.* We insure our buildings against fire, our autobiles against accident, and our lives against the hazards of kness and death.

The main purpose of every plan of insurance is to spread ongst a large number of persons a possible financial loss too ge to be borne conveniently by one person. This purpose accomplished by a large group of persons agreeing beforend to share the particular risk against which insurance is ired. Such a group is usually formed through an insurance npany.

Life insurance is largely a development of the last hundred rs. During this time it has become an institution of great portance not only to individuals to whom it furnishes urity from financial worries, but also to the governing lies and industries of the nation to which it contributes bility and safety. A large part of the savings of the people collected by insurance companies and invested by them in h-grade bonds, real estate mortgages, and some other types securities.

**Should I buy Life Insurance?** Let us consider an example the decisions which an individual has to make in the matter ife insurance. Robert Evans is a young man of 25, married l with a small child. He works for a regular salary but has had time to accumulate any considerable savings. When he t began to earn his living, he bought a policy providing 00 of life insurance, which he pays for by a part of his ings each year. Now, with more family responsibility l an increased salary, he decides to invest in another policy $10,000. In the event of his death, his wife, who is named his ***beneficiary,*** would receive from the insurance company ,000. Mr. Evans realizes that, in diverting a part of his ings regularly into these two insurance policies, he is buying ncial protection for his family and freedom from worry for nself.

**Some facts and technical terms.** The business of l insurance is carried on mainly by ***Life Insurance Compani*** of which some 41 operate in Canada under a licence from t Dominion Government. (Of these, 28 are Canadian co panies.) It is also carried on by a large number of fratern benefit societies. The contract between the Company a the person insured is called an ***insurance policy*** and t person insured is the policyholder. The payments which t policyholder makes for the insurance are called ***premium*** They may be paid yearly, monthly, or at other interva Occasionally a policy is paid for by a single premium at t time the policy is taken out. It is customary to quote t premiums for life insurance at so much per $1000 of insuran

Since life insurance is a protection against the risk death, which increases with advancing age, it follows that t premiums which a man pays depend on his age when he bou the policy. The greater his age, the greater are the premiu Before issuing a policy, the company requires evidence th the applicant is in good health. This evidence usually inclu a medical examination.

**What kinds of policies are there?** As you will find from the agents in a few years, there is a great variety in t kinds of policies offered by life insurance companies. companies, however, sell three main types, which are varied different provisions attached to the policies. These are kno as ***term insurance policies, whole life policies,*** and ***endo ment policies.***

**Term Insurance.** H. R. Smith, aged 30, invested all savings, as well as some borrowed money, in a business wh he had every reason to believe would be thriving in 10 ye To provide against the loss of his capital in case of his de within the next few years, he took out a term insurance pol for a 10-year period. He found that this policy would c him only $7.46 per $1000 each year. This is the chea

pe of insurance he could get, the reason being that, if he es for 10 years, the company has nothing to pay.

A term insurance policy usually contains a provision for nverting it into a more permanent type of insurance (whole or endowment). This would necessitate a larger premium. nan with a family of dependent children might decide that, the next few years, he needs more protection than he can ord on a permanent basis. He may then take out a term icy with the intention of converting it as soon as his cir- nstances make it possible for him to do so.

Another feature of many term insurance policies is that they ***renewable.*** In such case a policy contains a provision ereby it can be renewed for further periods. It might, for mple, be taken out for one year at a time and renewed each r. No further medical examination is required at the times renewal but the premium to be paid increases with each ewal to correspond with the increased age of the policy- der.

**Whole Life Policies.** In these policies, the insurance tinues throughout the life of the policyholder and the efit is payable at his death. There are several kinds of ole life policies, differing mainly in the manner in which premiums are paid. Those in which the premiums con- ue throughout the duration of the policy are called ***Ordinary e Policies*** (or ***Straight Life Policies***). Those in which premiums cease after a specified term of years are called ***nited Payment Life Policies.*** In the latter type, the niums are larger, since it is expected that there will be er of them.

The purpose of these policies may be illustrated by the ation of Robert Evans, already mentioned. For his 00 policy taken at age 21, he chose a 20-payment life policy, which the yearly premium was $25.45 per $1000. He will for this policy $50.90 a year for 20 years, after which the

policy will be fully paid up and the insurance will remain
force during his life. When he took out additional insuran
at age 25, Mr. Evans chose an Ordinary Life policy for whi
the premium was at the rate of $18.61 per $1000. Hence, f
this $10,000 policy he will make a yearly payment of $186.
for the remainder of his life.

**Endowment Policies.** An Endowment Policy is a co
bination of an investment with life insurance. Its use
illustrated in the following example.

David Campbell, on the birth of his son, decided to buy
20-year Endowment policy for $2000. If he should
within the next 20 years, the company will pay to his be
ficiary $2000. If he is living at the end of 20 years, he hims
will receive $2000. He feels more secure in the knowled
that a sum of $2000 will be available at the end of 20 ye
(and earlier if he should die before that time) for the educati
of his son or for setting him up in business. Mr. Campb
(aged 23) found the premium to be $46.69 per $1000, so t
his yearly payment is $93.38.

Endowment insurance is a relatively expensive type si
insurance is not the only benefit which is purchased. Besi
paying for insurance protection, the premiums must be su
cient to provide the amount of the endowment when the pol
matures.

**How are premium rates calculated?** The calculation
premium rates requires mathematical skill and training.
rates are based on probabilities shown by a ***Mortality Ta***
which is made up from past records of the number of pe
of a large group who died at different ages. The trai
expert who makes such calculations is known as an ***actua***
His profession is one of those which is open to students wit
mathematical training. The actuary must be sure that
premiums charged are sufficient for the following m

rposes: (*a*) to pay all claims which become due under the licies issued, and (*b*) to pay the expenses of operating the npany. In addition to these main items of expense, the emiums must also allow for unforeseen losses or expenses d for dividends to the shareholders, if any.

**Dividends to Policyholders.** Since the death rate among licyholders and also the interest earned on the company's rested funds cannot be predicted with certainty, the figure ich is arrived at for the premium, is to this extent, an imate. Should an effort be made to make this estimate respond as closely as possible to the actual cost of the urance? Two practices, which will now be described, are use by companies.

(1) On some policies, the rates charged are calculated so t the premiums are sufficient to meet all claims on the company l also to provide dividends to the policyholders, who thus ome participants in the company. Such policies are called ***rticipating.*** After a participating policy has been in force a few years, dividends are usually declared annually. These idends may be used by the policyholder in one of several ys. He may (*a*) withdraw the dividends in cash, (*b*) leave dividends with the company to accumulate at interest, (*c*) the dividends to purchase additional insurance, or (*d*) use dividends towards the payment of future premiums on his icy.

(2) On other policies, called ***non-participating,*** the npany charges somewhat lower premiums, which have been efully calculated, and pays no dividends on the policies. ch type of insurance has its merits. In practice, more urance is issued on the participating than on the non-ticipating basis.

**What choices has a policyholder in the use of his policy?** er the lapse of many years—30 or 40 for example—the gations of a policyholder may be very different from those

when he bought the policy. His family may be grown up an independent or he may have retired from business. He ma then wish to make a different use of his insurance from th stipulated in the policy. Most policies have a ***cash surrend value,*** which means that the policyholder can give up t policy in return for a cash payment. If he desires to disco tinue further premium payments, he can usually obtain paid-up policy for a reduced amount or have the policy co tinued in full for a limited term. Again, he may convert h policy into some other form,—an endowment into a paid-u life policy, for example. A possibility of some interest is use the proceeds from an endowment or the cash surrend value of a policy to purchase a ***life annuity,*** that is an annu payment for the remainder of his life. Another privilege of policyholder, which is useful in time of need, is that of borro ing from the company a sum of money (less than the ca surrender value), this sum to be repaid with interest later.

**What are Mutual Companies and Stock Companies?** The names refer to a difference in the manner of organizing insurance company. When a ***stock company*** is organized sum of money for the necessary expenses and stability of t enterprise is subscribed by individuals who receive ***sto*** certificates in return for their money and become shareholde in the company. Thereafter, the company pays interest dividends on this stock. A company which has no sha holders and no stock is called a ***mutual company.*** Such company consists of its policyholders and all profits on t company's operation belong to them.[2]

### EXERCISES

1. James Lane takes out a $5000 Ordinary Life policy $25.75 per $1000. What is his annual premium?

[2]Further information can be obtained from a pamphlet entitled: *Insurance, A Canadian Handbook* (The Macmillan Company of Can Limited, 1945).

2. If, on Mr. Lane's policy, dividends averaging 8% of the miums were paid each year by the company, what would his *net premium* for each year?

3. A man aged 60 pays an insurance company $13,333.33 return for an annual payment (or *life annuity*) of $1000. hat percentage of the purchase price does the annual pay- nt represent?

4. An employee bought a $5000 Ordinary Life policy and anged to pay for it by monthly deductions from his pay que. The premium was at the rate of $2.53 per month for 00 of insurance. If, at the end of the year, he receives a idend of $9.65, find the yearly cost of his insurance.

5. A 20-payment Life policy for a child aged 11 was bought a yearly premium of $18.88 per $1000. If the face of the icy was $2500, find the sum of all the premiums which will paid for the insurance. If the insurance were paid for in ance by a single premium, should this be more or less than sum of the yearly premiums? Why?

6. For an Ordinary Life policy for $5000 a man aged 25 s an annual premium of $86.50. At what rate is this mium quoted by the company? Would the annual premium more or less for a man aged 20?; for a man aged 30? Why?

7. Explain the difference between a 10-year Term Insur- e policy for $1000 and a 10-year Endowment policy for 00. If each of these policies is bought by a single premium, difference in cost would be approximately the present value 1000 due in 10 years. Explain the reason for this.

8. A man in middle life takes out a 20-payment Life policy $2000 at $36.82 per $1000. He deposits $500 with the pany to pay the first premium, which is due now, and er premiums as they become due. If the company pays %, compounded yearly, on deposits, show how the man's ount stands after the third premium has been paid.

9. Twenty years ago Mr. Landon took out a $2000 2
payment Life policy. The premium was $29.91 per $100
Now Mr. Landon finds that he has to his credit dividen
of $308 and that the cash surrender value of his policy
$1064. Find (*a*) the sum of the premiums paid, (*b*) the tot
amount he would receive if he were to surrender the polic
(*c*) the difference between these amounts. The differen
represents the amount by which the interest on the premiu
exceeds the cost of the insurance protection.

10. A $2000, 20-year Endowment policy requires premiu
at the rate of $47.25 per $1000. When the policy matur
dividends of $406 had accumulated to the credit of the poli
holder. By how much did the interest on the premiu
exceed the cost of the insurance?

## EXERCISES ON CHAPTER XIV

### *A*

1. Find the difference in the interest, compounded year
on (*a*) $100 for 1 year at 2% and (*b*) $100 for 2 years at 1%.

2. Find, without tables, the interest, compounded year
and the amount of: (*a*) $250 for 2 years at $2\frac{1}{2}\%$ per annu
(*b*) $75 for 3 years at 3% per annum.

3. Find, without tables, the interest, compounded h
yearly, and the amount of: (*a*) $100 for 1 year at $4\frac{1}{2}\%$
annum; (*b*) $460 for $1\frac{1}{2}$ years at 3% per annum.

4. Find the difference between the simple interest on $
at 4% per annum for 2 years and the compound interest
the same time and rate, if the interest is compounded h
yearly.

5. If $100 amounts to $134.39, find the present value
$100 for the same time and rate.

6. Find, without tables, the present value and the true scount, when interest is compounded yearly, of: (*a*) $340 2 years at 4% per annum; (*b*) $120 for 3 years at 5% per num.

7. Find, by the use of tables, the amount of: (*a*) $450 for years at 3% compounded yearly; (*b*) $625 for 8 years at 4% mpounded half-yearly.

8. Find, by the use of tables, the compound interest on: $358 for 11 years at 4% per annum, compounded yearly; $235 for 7 years at $3\frac{1}{2}$% per annum, compounded yearly.

9. Find, by the use of tables, the present value of: (*a*) 00 due 8 years hence at $3\frac{1}{2}$% per annum, compounded arly; (*b*) $500 due 20 years hence at 3% per annum, com- unded yearly.

10. A refrigerator, priced at $276, was bought on the tallment plan by a down payment of $100 and 3 monthly yments, the first two of $62 each and the third of $60.50. d the charge which the buyer paid for the privilege of ying by installments.

11. Mr. and Mrs. Forrest are considering the purchase of niture priced at $396. They are offered the privilege of ying for it in 15 monthly payments of $28.30, the first yment to be paid on purchase. Find the installment charge ich they would pay, on this plan.

12. Mr. Phillips, at the time of his retirement, finds that accumulated savings amount to $12,000, which is invested 3% per annum. He needs to withdraw $700 from his re- ves at the beginning of each year, for living expenses. If first withdrawal is made now, find how much of his capital remain just after the fifth withdrawal.

13. Mr. Phillips could provide for his old age by disposing his investments and using his capital to buy a government uity. This would guarantee him an income of $1200 a

year for life. Compare this method of providing for his ol age with that of the preceding exercise. What argument ca you think of in favour of each method?

**14.** A man, who is about to buy a $2000 life insuran policy, has to decide whether to take an Ordinary Life, a 2 payment Life, or a 20-year Endowment policy. Which these policies would require the largest annual premium ar which the smallest? On which would the policy remain force throughout his life? On which would the premiu have to be paid throughout his life?

**15.** If a man pays $50 a year for an insurance policy ar at the end of 20 years he receives $1000 as the cash surrend value of the policy, how was the cost of his insurance paid?

*B*

**16.** Make an estimate of the compound interest on $1 for 20 years at 4% per annum (*a*) compounded yearly, a (*b*) compounded half-yearly. Then, using the tables, c culate the two answers and find their difference.

**17.** By using the tables, find how many years it takes sum of money to double itself (*a*) at 2%, (*b*) at $3\frac{1}{2}$%, annum, compounded yearly.

**18.** A sum of $100 was invested for a child aged 10. Wh the child was 21, the amount to his credit was $124.47. interest was compounded half-yearly, find, by the use tables, the rate at which it was calculated.

**19.** Does a rate of 4% produce twice as much sim interest as a rate of 2%? Does a rate of 4% produce tw as much compound interest as a rate of 2%? If not, doe produce more or less than twice as much?

**20.** Each of two children, aged 8 and 13 respectiv receives a bequest of $1000, to be paid to him when he reac

ıe age of 21. If money is worth $2\frac{1}{2}\%$ per annum, compounded :arly, find from the tables the present value of each child's :quest.

**21.** Alfred needs a bicycle for his paper route and finds ıe cash price to be $61. He can pay $25 in cash and is fered the privilege of completing the payment in 3 install- ents of $13 each, spread out at intervals of 4 months. Cal- ılate the rate of interest which he would pay, on the install- ent plan.

**22.** The cash price of a refrigerator is $250. Which of the llowing two offers is more advantageous to the customer? ) $100 down and $165 at the end of a year; (*b*) $100 down, ;1 at the end of 6 months and $81 at the end of a year. For ch offer, calculate the rate of interest which the customer ɔuld pay.

**23.** The cash price of an electric stove is $160. It is ɔught on the installment plan with a down payment of $45. ı which of the following plans for completing his payments ɔuld the buyer be paying a smaller rate of interest? Cal- ılate each rate. (*a*) Four payments of $30 each at intervals 3 months; (*b*) Three payments of $40 each at intervals of 4 ɔnths.

**24.** Suppose you are buying a $60 radio on which you are le to make a cash payment of $20 and you wish to pay the mainder in 4 installments at intervals of 3 months. (*a*) ;timate what you think would be a fair amount for each ;tallment. (*b*) If each installment were $11, what rate of :erest would the installment charge of $4 represent?

**25.** Mr. B. borrowed $200 from his bank, to be repaid on e following installment plan. The bank added an interest arge of $6.50, the resulting amount to be repaid in 12 equal ɔnthly installments. (*a*) Calculate the amount of each ;tallment. (*b*) Calculate the rate of interest which Mr. B. ys.

**26.** Mr. C. obtained a loan from his bank on a slightly different plan. He received \$193.72 from the bank and he repaid a total of \$200 in 12 equal monthly payments. Calculate the rate of interest which he paid.

**27.** A man at age 30 is deciding whether to buy an Ordinary Life insurance policy at \$21.21 per \$1000 or a 20-payment life policy at \$30.13 per \$1000. If he should die within the next 20 years, which policy would prove to have been the more advantageous? If he were sure of living to an advanced age which type would he be likely to choose? Why?

**28.** Mr. Kean bought a \$1000 20-year Endowment policy for an annual premium of \$48.55. When the policy matured he received the face of the policy and dividends of \$25[illegible]. Find the sum of all the premiums which he paid. Find the amount, at the time the policy matured, of the first 2 premiums reckoning interest at $3\frac{1}{2}\%$, compounded yearly.

**29.** Mr. Sterne bought a life insurance policy requiring premiums of \$52.40 a year. He deposited \$200 with the company to pay premiums as they became due and directed that any dividends that were declared on his policy should be added to his deposit. At the end of each of the second and third years, a dividend of \$8.60 was declared. If the company paid interest on deposits at 3% per annum, compounded yearly, find the amount in Mr. Sterne's deposit account after the fourth premium had been paid.

**30.** Using the income tax rates given on page 300, calculate the tax which would be paid by a man with a family of 4 children, whose taxable income is \$3150.

*C*

**31.** In the table of amounts (page 407), the highest rate given is 6%. Calculate the first 5 entries that would be needed for a column of amounts at 8%.

**32.** Calculate the first 5 entries that would be needed for column of present values of \$1 at 8%.

**33.** Find, from the tables, in how many years a sum of ıoney at 2% would amount to as much as it would in 25 years t 3%, compounded yearly.

**34.** If interest is calculated at 4% per annum, compounded times a year, find the amount of \$500 for 8 years.

**35.** If interest is calculated at 3% per annum, compounded ılf-yearly, find the present value of \$400 due 6 years hence.

**36.** Calculate, from the tables, the difference between the ɔmpound interest on \$1000 for 20 years at 3% per annum if ıe interest is compounded yearly, and that if it is compounded ılf-yearly.

CHAPTER XVI

# THE USE OF TRIGONOMETRIC RATIOS

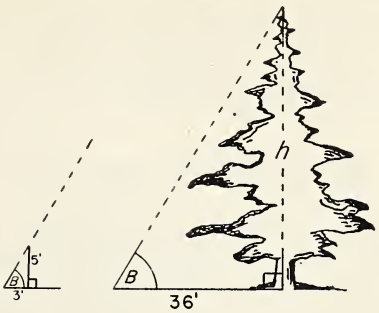
## *Shadow Reckoning*

Mr. Allen, who is 6 feet tall, noticed, when they wer walking together, that his shadow was twice as long as tha of his son David. How tall is David?

The shadow of a garage was found to be one-quarter a long as that of a nearby barn. How does the height of th garage compare with that of the barn?

A fence-post 5 feet high casts a shadow 3 feet long at th same time as a pine tree casts a shadow 36 feet long. Fin the height of the tree.

It is not hard to see why ***shadow reckoning*** came to b used long before the invention of accurate measuring instru ments. One of the stories that has come down to us dates fro about 600 B.C. It records that Thales, known as one of th seven wise men of ancient Greece, astonished the Egyptian when, on a visit to their country, he calculated the heights the pyramids by measuring shadows.




Let us look more closely at the problem, posed above, of nding the height of the pine tree. A right-angled triangle is rmed by the fence-post, its shadow, and the ray of sunlight om the top of the post to the end of the shadow. A second ght triangle is formed similarly by the tree and its shadow. nce the rays of sunlight have all the same direction, the two ght triangles are equiangular. Now you have already arned that triangles which are equiangular have the same iape, or they are ***similar.*** A side in one of the two similar iangles is said to ***correspond*** to a side in the other when ie two sides are opposite equal angles.

We now remind you of a property of two similar triangles page 205). It is the property used in shadow reckoning.

> ***In two similar triangles, the ratio of any two sides of one is equal to the ratio of the corresponding sides of the other.***

et us apply this property to the calculation of the height of ie tree. The ratio $h/36$ from the pine-tree triangle is equal

to the ratio 5/3 from the fence-post triangle. Since

$$h/36 = 5/3, \text{ therefore } h = 60,$$

and the height of the tree is 60 feet.

## *We Introduce a New Name*

Let us place a label $B$ on each of the equal acute angles a the bases of the triangles in the preceding figures. Suppos that we know about the two triangles only that they ar right-angled and that the angles marked $B$ are equal. Sinc two of the angles in the small triangle are equal to two of thos in the large triangle, the triangles are similar (Ex. 2, page 207) It follows that the ratio of any two sides of one is equal to th ratio of the corresponding sides of the other.

This last is a striking result and deserves more study. Let u put it in another way. ***One*** of the acute angles in a righ triangle is sufficient to determine the shape of the triangl completely and, hence, to determine all the ratios of its thre sides.

The ratio which was used in finding the height of the tre was $h/36$; this we may describe as

$$\frac{\text{the side opposite } B}{\text{the side adjacent to } B}.$$

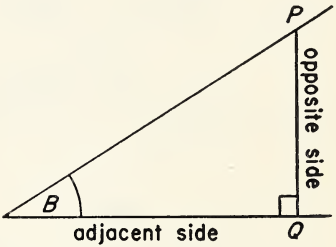
(The third side is also adjacent to $B$ but, as it is opposite th right angle in the triangle, we shall avoid confusion by callin it the hypotenuse.) This ratio which, we repeat, depends fo its value on $B$ is called the ***tangent*** of $B$; because of it frequent use it is abbreviated to ***tan B.*** In the figure w have

$$\tan B = h/36 = 5/3.$$

You may be curious about this use of the word *tangent*, which has tw different meanings in mathematics. Its use to stand for a ratio as her

efined dates from ancient times. For the present we may regard the names ngent, sine, and cosine (you will soon learn the last two) much as we do ie names John, Fred, and Dave. The boys' parents may have had good asons for naming them as they did but we seldom inquire into these reasons. milarly, we need not inquire why our mathematical ancestors chose some the names they did.

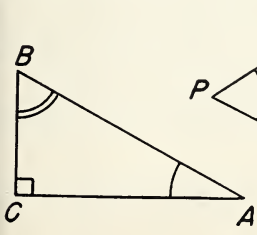
Now let an acute angle be given. From any oint $P$ on one of its arms raw a perpendicular $PQ$ to ie other arm. This forms right-angled triangle hose shape is determined y the given angle. In iis right triangle,

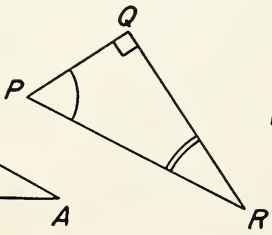


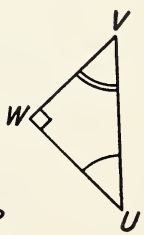

*the tangent of the angle B* (*or* **tan B**) $= \dfrac{\textit{the side opposite}}{\textit{the side adjacent}}$

## EXERCISES

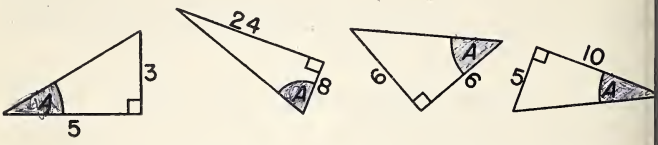
**1.** In each of the following figures write the tangent of each the acute angles.



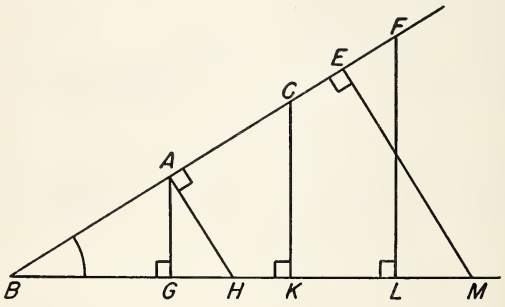



2. Give the numerical value of tan $A$ in each of the following figures.




3. From the following figure write the tangent of $B$ in different ways.

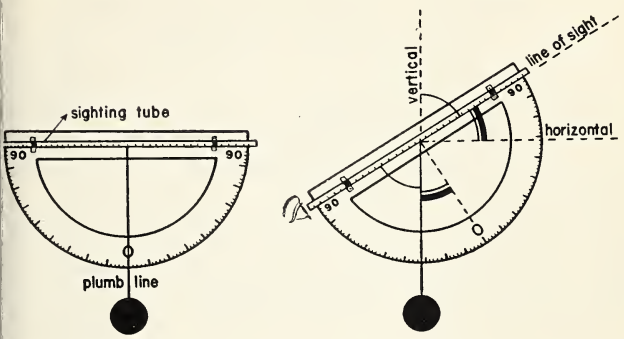



## *A Digression on Angles*

**Angles of elevation and of depression.** When Mr. Bates who studies astronomy, told us to look for a particular star in the southwest, he said that it was about 45° above the horizon. It is easy to see what this means. Another way of saying it is that the angle of elevation of the star was about 45°. To find the angle of elevation of any object, a cloud, a balloon, or the top of a flagpole, for example, surveyors use an elaborate instrument called a ***transit***.

You can make a less accurate instrument of your own by attaching a hollow sighting tube and plumb-line to a good

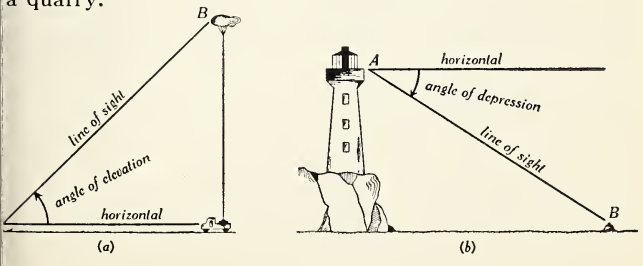
ed protractor, as shown in the figure. By sighting the ject through the tube, you can easily find the angle between




e horizontal and the line of sight. (Show by geometry that is angle is equal to that between the zero-line and the plumb-e.)

> *The angle measured upward from the horizontal to the line of sight through the object is the* ***angle of elevation.***

Consider now an object below the observer's horizon, such a building observed from a balloon, or a rock at the bottom a quarry.



*The angle measured downward from the horizontal to the line of sight through the object is the* ***angle of depression.***

**Units for measuring angles.** For our fundamental un in angle measurement we take a ***right angle,*** the angle betwe two perpendicular lines. Right angles occur so commonly buildings and other structures that it is important for us recognize their size and to detect when an angle differs, ev slightly, from a right angle. Since a right angle is relative large, it is divided into 90 equal parts called ***degrees.*** Y can see the size of a degree by looking at your protract Another way of becoming used to its size is by noting that t two lines from your eye to the opposite edges of a full mo make an angle of approximately half a degree. It would ta slightly fewer than 360 full moons to reach from the easte horizon across the sky to the western.

When very small divisions of an angle are needed, the degr is subdivided into 60 equal parts called ***minutes.*** Wh you reflect that the angle subtended at your eye by the moo diameter is about 31 minutes, you will realize how small minute is. It is smaller than is needed for the applicatio which most of us have to make. However, specialists w need very precise measurements make a further division the minute into 60 ***seconds.*** A second is indeed a sm angle. If two lines from your eye make an angle of o second with each other, then at a distance of 3 miles they a less than an inch apart!

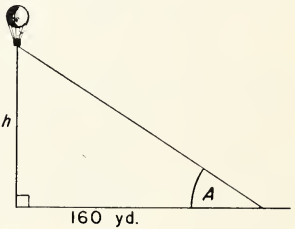
$$1 \text{ right angle} = 90^\circ$$
$$1^\circ = 60'$$
$$1' = 60''$$

## CAUTION

*Do not be confused by the fact that minutes and seconds are used as units of angle as well as of time. It causes no confusion that the word pen is sometimes associated with chickens and sometimes with ink. Similarly, in any problem in which the term minute is used, you will have no difficulty in deciding whether time or angle is meant.*

*ample* 1. A balloon is directly above a certain point on e ground at which it is tethered. When a man walks 160 rds directly away from this point, he finds the tangent of the gle of elevation of the balloon to be 0.65. Find the height the balloon.

the required height is $h$, we ve $h/160 = \tan A = 0.65$. ence $h = 0.65 \times 160 = 104$. ace 0.65 is correct only to 2 ures, the answer is correct ly to 2 figures. In other words, e balloon is about 100 yards ove the ground.




### EXERCISES

1. The altitude of an isosceles triangle, drawn from the rtex in which the equal sides meet, is 10 inches long and the ird side of the triangle is 10 inches long. What is the tangent each of the equal angles?

2. In a rectangle 14 inches by 35 inches a diagonal is drawn. nd the tangents of the angles which the diagonal makes with e sides.

3. Draw two lines at right angles and from the inter- tion $A$ cut off lengths $AB$ and $AC$ of 1 inch and 2 inches spectively. Join $BC$. What is the tangent of the angle $B$

and of the angle $C$? Construct similarly an angle whos tangent is 3, and one whose tangent is $\frac{2}{3}$. (Consider thes data exact.)

4. After three members of the family had given the estimates of the height of the peak of their barn, Evere undertook to calculate it. Using a tape line, he found th distance of the tip of the shadow from the base of the bar directly below the ridge to be 26 feet. He then set up yardstick vertically between two bricks and found its shado to measure 21 inches. What should he find for the height the barn?

5. A telegraph pole has an iron stirrup 40 inches from th ground. When the sun was shining, it was found that th shadow of the stirrup was 15 inches from the base of the po and that the length of the shadow of the pole was $13\frac{1}{2}$ fee Find the height of the pole.

6. If a flagpole casts a shadow 48 feet long when the sun rays are striking the ground at an angle whose tangent is what is the height of the pole?

7. A man standing on a cliff 80 feet above the level of lake observes that the tangent of the angle of depression of boat is 0.35. Find the distance of the boat from the foot the cliff.

8. In order to find the height of a tall oak tree standing level ground, a convenient point was chosen from which th top of the tree could be observed. The tangent of the ang of elevation was found to be 1.25 and the distance of the poi of observation from the trunk of the tree was found to be 1 feet. Calculate the height of the tree.

9. John's estimate of the height of a tall smoke-stack w 200 feet and Fred's estimate was 250 feet. They then decid to make an indirect measurement of the height. For th purpose they measured a distance of 132 feet from the ba of the stack. Then, with their home-made instrument, th

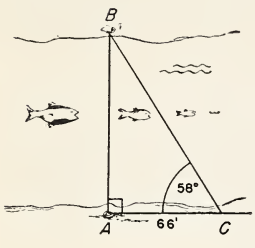
served the angle of elevation $A$ of the top and found that
n $A = 1.7$. With these measurements how accurately
uld they find the height of the stack and what answer should
ey give?

**10.** From the bow of a ship the look-out has an angle of
vation with a tangent of 5/6. From the look-out the
rn of the ship has an angle of depression with a tangent of
/35. The mast on which the look-out is situated is 72
t from the bow. Find (*a*) the height of the look-out above
e deck, (*b*) the total length of the ship.

## *Trigonometric Tables*

You have now seen the usefulness of tangents of angles for
e calculation of heights and distances which are not easily
easured directly. This raises the practical question of how
ese tangents are obtained. With a protractor or some more
curate instrument, it is possible to measure the angles in
grees and minutes. Knowing the size of an angle, how
uld you find its tangent? * * * Here is an
vious method: construct the angle on your paper by means
a protractor, form a right triangle as described above,
easure the opposite and adjacent sides, and take their ratio.
the solution of a problem by means of a tangent, the cal-
lation of the tangent is generally the most laborious part.
rtunately for us, however, tangents have been used for so
ig a time by engineers and other computers that they have
n the advantage of having this work done once and for all.
ey have, in fact, devised more economical ways of calculat-
g tangents and have recorded their values in tables for our
nvenience. A short trigonometric table is given on page 337.
e following example illustrates the use of the table.

*Example* 2. To find the width of the Cataract River, som boys in Grade X selected a stum at $A$ on one bank and sighte directly across the river to a ro $B$. They then laid off a right ang at $A$ and measured a distance of 6 feet along $AC$. Next, they foun the angle $ACB$ to be 58°. The calculation was then as follows:




$$AB/AC = \tan 58°.$$

Hence $AB = AC \tan 58° = 66 \times 1.600 = 105.6$, and th width of the river is 106 feet, approximately.

Note on the approximations in this solution. If the 66-foot distance measured correct to one-tenth of a foot, and the 58° angle to one-tenth of degree, then the data are said to be correct to 3 figures and the answer 106 is justified. If, however, the data are correct only to the nearest foot a the nearest degree, respectively, then the answer should be reported as 1 feet, correct to 2 figures.

## EXERCISES

1. In order to find the height of the school building, Jo placed his instrument against a tree in the yard and adjust his line of sight to the peak of the roof. The angle of elevati was 36°. He then measured the distance from the tree to point at the base of the building directly below the peak a found it to be 85 feet. His instrument was 3 feet above t ground. Find the height of the building.

2. A farmer wishes to find the distance across a gull which he cannot easily cross. Make a diagram to show h he might proceed if he had a tape line, an instrument for measu ing angles, and a table of tangents.

3. The shadow of a steeple was observed to be 65 feet lo when the angle of elevation of the sun was 62°. Find t height of the steeple.

| *grees* | *sine* | *cosine* | *tangent* | *degrees* | *sine* | *cosine* | *tangent* |
|---|---|---|---|---|---|---|---|
| 0 | .00 | 1.00 | .00 | 45 | .71 | .71 | 1.00 |
| 1 | .02 | 1.00 | .02 | 46 | .72 | .69 | 1.04 |
| 2 | .03 | 1.00 | .03 | 47 | .73 | .68 | 1.07 |
| 3 | .05 | 1.00 | .05 | 48 | .74 | .67 | 1.11 |
| 4 | .07 | 1.00 | .07 | 49 | .75 | .66 | 1.15 |
| 5 | .09 | 1.00 | .09 | 50 | .77 | .64 | 1.19 |
| 6 | .10 | .99 | .11 | 51 | .78 | .63 | 1.23 |
| 7 | .12 | .99 | .12 | 52 | .79 | .62 | 1.28 |
| 8 | .14 | .99 | .14 | 53 | .80 | .60 | 1.33 |
| 9 | .16 | .99 | .16 | 54 | .81 | .59 | 1.38 |
| 10 | .17 | .98 | .18 | 55 | .82 | .57 | 1.43 |
| 11 | .19 | .98 | .19 | 56 | .83 | .56 | 1.48 |
| 12 | .21 | .98 | .21 | 57 | .84 | .54 | 1.54 |
| 13 | .22 | .97 | .23 | 58 | .85 | .53 | 1.60 |
| 14 | .24 | .97 | .25 | 59 | .86 | .52 | 1.66 |
| 15 | .26 | .97 | .27 | 60 | .87 | .50 | 1.73 |
| 16 | .28 | .96 | .29 | 61 | .87 | .48 | 1.80 |
| 17 | .29 | .96 | .31 | 62 | .88 | .47 | 1.88 |
| 18 | .31 | .95 | .32 | 63 | .89 | .45 | 1.96 |
| 19 | .33 | .95 | .34 | 64 | .90 | .44 | 2.05 |
| 20 | .34 | .94 | .36 | 65 | .91 | .42 | 2.14 |
| 21 | .36 | .93 | .38 | 66 | .91 | .41 | 2.25 |
| 22 | .37 | .93 | .40 | 67 | .92 | .39 | 2.36 |
| 23 | .39 | .92 | .42 | 68 | .93 | .37 | 2.48 |
| 24 | .41 | .91 | .45 | 69 | .93 | .36 | 2.61 |
| 25 | .42 | .91 | .47 | 70 | .94 | .34 | 2.75 |
| 26 | .44 | .90 | .49 | 71 | .95 | .33 | 2.90 |
| 27 | .45 | .89 | .51 | 72 | .95 | .31 | 3.08 |
| 28 | .47 | .88 | .53 | 73 | .96 | .29 | 3.27 |
| 29 | .48 | .87 | .55 | 74 | .96 | .28 | 3.49 |
| 30 | .50 | .87 | .58 | 75 | .97 | .26 | 3.73 |
| 31 | .52 | .86 | .60 | 76 | .97 | .24 | 4.01 |
| 32 | .53 | .85 | .62 | 77 | .97 | .22 | 4.33 |
| 33 | .54 | .84 | .65 | 78 | .98 | .21 | 4.70 |
| 34 | .56 | .83 | .67 | 79 | .98 | .19 | 5.14 |
| 35 | .57 | .82 | .70 | 80 | .98 | .17 | 5.67 |
| 36 | .59 | .81 | .73 | 81 | .99 | .16 | 6.31 |
| 37 | .60 | .80 | .75 | 82 | .99 | .14 | 7.12 |
| 38 | .62 | .79 | .78 | 83 | .99 | .12 | 8.14 |
| 39 | .63 | .78 | .81 | 84 | .99 | .10 | 9.51 |
| 40 | .64 | .77 | .84 | 85 | 1.00 | .09 | 11.43 |
| 41 | .66 | .75 | .87 | 86 | 1.00 | .07 | 14.30 |
| 42 | .67 | .74 | .90 | 87 | 1.00 | .05 | 19.08 |
| 43 | .68 | .73 | .93 | 88 | 1.00 | .03 | 28.64 |
| 44 | .69 | .72 | .97 | 89 | 1.00 | .02 | 57.29 |
| 45 | .71 | .71 | 1.00 | 90 | 1.00 | .00 | ..... |

4. The chimney of the International Nickel Company
port Colborne is 512 feet high. How long a shadow will
cast when the sun is at an elevation of 35° above the horizo

5. From two observation points 2460 yards apart observe
who are in telephone communication with each other, a
watching an aeroplane. The plane was reported to
directly above one station at the same time as it had an an
of elevation of 42° from the other. At what height was t
plane flying?

6. To find the height of a tall tree, Stanley measured t
length of its shadow and found it to be 165 feet. He th
observed the upper edge of the sun just over the top of t
tree and found its angle of elevation to be 39°. Find t
height of the tree.

7. A boy who was standing 42 feet from a telegraph p
and using an instrument 4 feet above the ground found t
angle of elevation of the top of the pole to be 51°. Find t
height of the pole.

8. At what angle is a ladder inclined to the vertical wh
it rests on the ground 6.0 feet from the wall of a house a
reaches a window 22 feet above the ground?

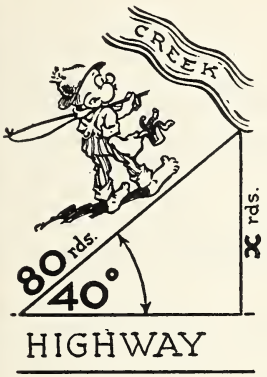
9. A cylindrical well is 6.0 feet in diameter. From a poi
on the edge, it is observed that the angle of depression of t
water's edge on the opposite side from the observer is 7
Find the depth to the surface of the water.

10. A house stands 320 feet from a north-and-south roa
A man walking on the road observes the house to bear 22° e
of south. Find the man's position on the road.

## *Two More New Names*

A boy with a fishing rod left the highway on which he w
walking and proceeded on a path at 40° with the highway f

listance of 80 rods, when he reached a creek. How far was then from the highway?



An 18-foot ladder leans against the wall of a house, making angle of 60° with the ground. How far is its foot from the ıll?

In order to solve these two problems, let us recall the portant fact that in a right triangle ***one*** of the acute angles sufficient to determine the shape of the triangle and hence e ratios of its sides. In the first figure the measure of the quired distance is $x$ and the ratio which suffices to find it is 80. This ratio is determined by the angle 40°; it is called e ***sine*** of 40°. For convenience the relation is abbreviated, us:

$$x/80 = \sin 40^\circ \text{ (read: sine } 40^\circ\text{)}.$$

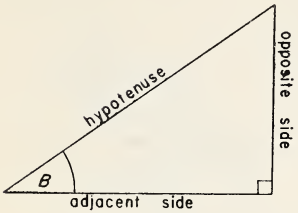
ıen

$$x = 80 \sin 40^\circ.$$

In the second problem, the distance of the foot of the ladder ɔm the house is shown in the figure as $y$ feet. This is found ɔm the ratio $y/18$, which is called the ***cosine*** of 60°. The lation is written for short

$$y/18 = \cos 60° \text{ (read: cosine 60°).}$$

Hence, $$y = 18 \cos 60°.$$




Now refer again to th angle $B$ and the right triang which was constructed for th definition of the tangent.

***Definition of the sine and the cosine***

*The sine of B* (*or* ***sin B***) $= \dfrac{\textit{the side opposite}}{\textit{the hypotenuse}}$

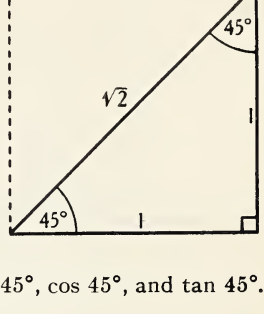
*The cosine of B* (*or* ***cos B***) $= \dfrac{\textit{the side adjacent}}{\textit{the hypotenuse}}$

The sine, cosine, and tangent of an angle are called ***trig nometric ratios*** of the angle. These ratios and their recipr cals form the basis for the study of ***trigonometry*** (Gree *trigon*, triangle; *metron*, measure).

The calculation of the approximate values of the trigon metric ratios in decimal form is, for most angles, a laborio process. Fortunately, it has been done for us by the m who made the tables. It is interesting to notice, howev that there are certain specified angles for which it is easy calculate the trigonometric ratios.

## *The Trigonometric Ratios of Some Specified Angles*

First let us take an isosceles right triangle, one of th triangles which is formed by drawing the diagonal of a squar




ıe angles of this triangle are °, 45°, and 90°. Since the size each angle remains the same matter how large or small the angle is, it is convenient to ke a triangle in which each of e equal sides is one unit long. ıen the ever-useful theorem of thagoras is waiting to tell us e length of the hypotenuse; it $\sqrt{1^2 + 1^2}$ or $\sqrt{2}$ units. Now ıd from the figure the values of sin 45°, cos 45°, and tan 45°.

* * *

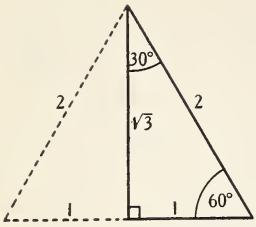
re are the numbers which you have just found.

$$\sin 45° = 1/\sqrt{2}, \quad \cos 45° = 1/\sqrt{2}, \quad \tan 45° = 1.$$

we wish to express the first two of these numbers in decimal m, we notice that $1/\sqrt{2} = \sqrt{2}/2$ and from a table of square ots we find that $\sqrt{2} = 1.414$. Hence $\sin 45° = \cos 45° = 14/2 = 0.707$, correct to 3 figures.

The isosceles right triangle, which gave us the ratios of 45°, is animportant pe and one which we should fix in our memories. Observe this shape as it urs in buildings, machines, and decorative designs. Draw several isosceles ıt triangles of different sizes in different positions, for example with the otenuse horizontal and with it vertical. Notice that the hypotenuse is ays $\sqrt{2}$ times as great as either of the other two sides. This is also ex- ssed by saying that the three sides have to one another the ratios of 1 to $\sqrt{2}$. If one of the equal sides has length $a$, the lengths of the three sides $a$, $a$, and $a\sqrt{2}$.

The half square gave us the trigonometric ratios of 45°. t us now take half an equilateral triangle. By drawing the rpendicular from one vertex to the opposite side, we get a ht triangle with angles 30°, 60°, and 90°.




Since the shortest side of the 30° 60°-90° triangle is one-half the hypotenuse, it is convenient to take as its length 1 unit, in which case the hypotenuse is 2 units. We now call upon the Pythagorean theorem to give us the third side; its answer is $\sqrt{2^2 - 1^2}$ or $\sqrt{3}$ units. Now you can read the trigonometric ratios of 60° and those of 30° from the same triangle.

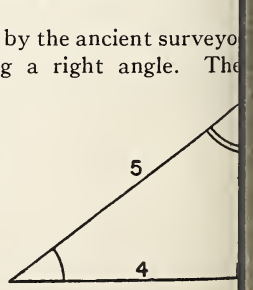
* * *

$$\sin 60° = \sqrt{3}/2, \quad \cos 60° = 1/2, \quad \tan 60° = \sqrt{3},$$
$$\sin 30° = 1/2, \quad \cos 30° = \sqrt{3}/2, \quad \tan 30° = 1/\sqrt{3}.$$

The triangle with angles 30°, 60° and 90° is another shape which combines beauty with utility. Practise drawing it in various positions. Your test of the figure is that it should be half an equilateral triangle. Its sides have to one another the ratios of 1 to 2 to $\sqrt{3}$. If the shortest side is $a$, the hypotenuse is $2a$ and the third side is $a\sqrt{3}$.

## *The 3—4—5 Triangle*

You have learned of a method used by the ancient surveyors along the river Nile for constructing a right angle. They formed a loop of rope 12 units long with knots at intervals of one unit. This rope was stretched in the form of a triangle whose sides were 3, 4, and 5 units. You will now be able to prove that this triangle is right-angled. The trigonometric ratios of its two acute angles are easily found.



## EXERCISES

1. From a table of square roots you find that $\sqrt{3} = 1.732$ proximately. Hence find in decimal form the trigonometric ios of 30° and of 60°.

2. Construct a 3-4-5 triangle, name its two acute angles, d find their trigonometric ratios.

3. In a Maypole dance a string 15 feet long is attached to e top of the Maypole and held by a girl at an angle of 60° th the pole. How far is the girl from the pole? If she holds e string 3 feet above the ground, how high is the pole?

4. In an isosceles right triangle the length of each equal e is $2\frac{1}{2}$ inches. Find the length of the hypotenuse.

5. In an isosceles right triangle the length of the hypotenuse 12 inches. Find the length of the other two sides.

6. In a 30°-60°-90° triangle the length of the shortest side 5 cm. Find the lengths of the other sides.

7. In a 30°-60°-90° triangle the side opposite the 60° angle 8 inches. Find the lengths of the other two sides.

8. Two rafters, each 21.0 feet long, meet at right angles the peak of a roof. If each rafter projects 1.0 foot beyond e wall on which it rests, find the distance between the walls.

9. A ladder 18 feet long leans against the wall of a barn th its foot 9 feet from the wall. Find (*a*) the angle which e ladder makes with the ground, (*b*) the height of the top of e ladder above the ground.

10. A traveller, asking for directions, is told that Wood- idge is 12 miles away "as the crow flies", in a direction 23° st of north. To reach Woodbridge how far east will the aveller have to go and how far north?

11. A guy wire on a telegraph pole is fastened to the ground feet from the pole and makes 52° with the ground. Find

(*a*) the length of the guy wire, (*b*) the distance from the grou to the point where it is attached to the pole.

**12.** A pump handle 3 feet long is attached to a pump 4 f above the platform of the well. In its highest position t handle is horizontal. It moves through an angle of 50° to lowest position. How far then is the end of the handle fr (*a*) the pump, (*b*) the platform?

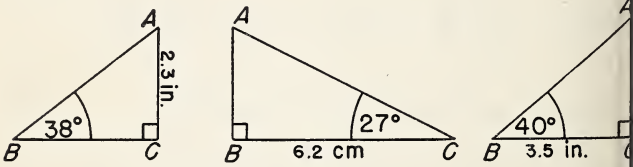
**13.** A cone 18 inches high has a vertical angle of 42°. Fi (*a*) the radius of its base, (*b*) its slant height.

**14.** A flower basket 15 inches in diameter at the top suspended from a hook by strings 15 inches long attached the rim of the basket. What angle do the strings make wi (*a*) the vertical, (*b*) the horizontal?

## EXERCISES ON CHAPTER XVI

### *A*

**1.** Find the length of $AB$ in each of the following rig triangles:




**2.** Draw an angle of 36°. Find, by measurement and c culation, its sine, cosine, and tangent. Using the tabl check the accuracy of your work.

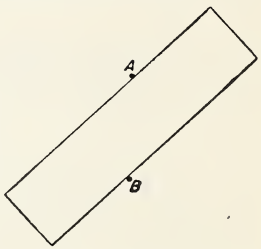
**3.** (*a*) A steep road rises 1 foot in every 10 feet measur horizontally. What is the tangent of the angle of inclination the road? (The angle of inclination is the angle which t road makes with the horizontal.) (*b*) A road rises 1 foot every 10 feet measured along the road. What is the sine the angle of inclination of the road? Find the angle to t

rest degree. Which of the two roads of (*a*) and (*b*) is eper?

**4.** The face of an embankment at the side of a road measures feet up the slope. If the top of the embankment is 16 feet ve the road, at what angle does the face meet the road?

**5.** A boy is flying a kite at the end of a 200-foot string. If string has no sag, and makes an angle of 65° with the ground, l the distance from the boy to the point on the ground ctly under the kite. Neglect the height of the boy.

**6.** A rectangular strip of wood is ported at an angle of 45° with the izontal by two pegs $A$ and $B$. $A$ vertically above $B$, and $AB = 3$ hes. Find the width of the strip.




**7.** At a turn in the road, a telephone pole is supported by a y wire which is fastened to the pole 20 feet above the ground l to the ground 12 feet from the foot of the pole. Find the gle which the wire makes with the pole and the angle with ground.

**8.** A road up a hill rises at 8° to the horizontal. When a d is drawn 200 feet up the hill, through what vertical tance is it raised?

**9.** The Eiffel Tower in Paris is 980 feet high. Find the gth of its shadow when the sun is 63° above the horizon.

**10.** A ladder 40 feet long just reaches to the roof of a house en it makes 60° with the ground. Find the height of the f above the ground.

**11.** From a point 90 feet from its base, the angle of elevan of the top of a monument is 61°. Find the height of the nument.

**12.** A ladder 36 feet long has its upper end at the top of a vertical wall and its lower end on level ground 15 feet from the base of the wall. Find (*a*) the angle between the ladder and the ground, (*b*) the height of the wall.

**13.** A tunnel in a mine is constructed at an angle of with the horizontal. When a miner has followed the tunnel for $\frac{1}{4}$ mile, how far is he below the level at which he started?

**14.** A wedge measures 8 in. along the sides from tip to base and is $2\frac{1}{2}$ in. thick at the base. Find the angle at the point of the wedge.

**15.** An aviator, flying at a height of 4200 feet, observes the angle of depression of his airport to be 25°. How far is the airport from the point on the ground directly below him?

*B*

**16.** A wire fastened to a picture at the points $B$ and $C$ passes over a nail at $A$. If $BC = 18$ inches and $AB = AC = 15$ inches, find the sine, cosine, and tangent of the angle $B$.

**17.** A flight of stairs has a banister along the side. Each step is 9 inches wide and each riser is 6 inches high. Find the angle of inclination of the banister-rail.

**18.** From a point on an east-and-west road, the bearing of a silo was N 38° E. From a point 1200 feet farther east on the road, the bearing was N 52° W. Find the distance from the second point to the silo.

**19.** A swing is supported by two hooks in a horizontal beam. The distance from each hook to the seat is 15 feet, and the seat is 2 feet above the ground. When the swing is turned through an angle of 30° from its stationary position, how far is the middle point of the seat above the ground?

**20.** A surveyor's instrument is supported on a tripod with 3 legs each $5\frac{1}{2}$ feet long. If each leg is inclined at 72° to the horizontal, find the distance of the bottom of the instrument above the ground.

**21.** To avoid a level crossing, a highway is built up so as to cross above a railway track, the distance of the highway above the track being 20 feet. On each side of the bridge the highway slopes at 5° to the horizontal. How far from the track was it necessary to begin making the fill for the bridge?

**22.** A heavy downpour of rain was driving from the west at an angle of 20° with the vertical. A pedestrian took shelter on the east side of a building 32 feet high. If the pedestrian is $5\frac{1}{2}$ feet high, how close to the building must he stand in order to avoid the rain?

**23.** The ridge-pole of a roof is 9 feet above the centre of the attic floor and the attic is 36 feet wide. If the roof meets the edge of the attic floor, find its angle of inclination.

**24.** Engineers sometimes say that a road has a 3% slope when it rises 3 feet in a horizontal distance of 100 feet. What is the tangent of the angle of inclination of such a road? What is the tangent of the angle of inclination of a roof which has a 30% slope?

**25.** For the purpose of night flying, observations are made to determine the 'ceiling' or the height of the clouds. In one experiment, a searchlight sent out a vertical ray of light which illuminated a cloud. This illumination was observed from a point on the ground 3000 feet from the searchlight and the angle of elevation was found to be 32°. What 'ceiling' should the observer report?

**26.** At a later time, the observer of the preceding exercise reported a ceiling of 1500 feet. What angle of elevation did he observe?

*C*

**27.** An observer in an aircraft, when it was vertically above a farmhouse, noted that the angle of depression of the foot of a monument was 30°. The distance from the house to the monument was 0.75 miles. If the house and the foot

of the monument are 200 feet above sea level, what is the altitude of the aeroplane above sea level?

**28.** In a strong wind, a flagstaff was broken. The two parts, however, were not separated. The top of the pole rested on the ground at a point 10 feet from the foot, and the angle between the fallen upper part and the ground was 58°. Find the original height of the flagstaff.

**29.** $M$ and $N$ are two points 200 feet apart on the straight bank of a river. $P$ is a point on the opposite bank such that $\angle PMN = 45°$ and $\angle PNM = 30°$. Calculate the width of the river.

**30.** A pavement has an angle of inclination of 5° from a point $A$ to a point $B$. $A$ and $B$ are 2000 feet apart, measured horizontally. The pavement is level from $B$ to a point $C$, 100 feet distant from it. If the pavement had had a uniform inclination from $A$ to $C$, what would have been its angle?

**31.** A 30-foot ladder is raised while its foot is kept stationary on the ground. Compare the angle of inclination of the ladder when the top is 20 feet above the ground with that when the top is 10 feet above the ground. Is the first-named angle twice the first, or more, or less? Make your guess first, then calculate the angles.

**32.** A flagpole stands on the top of the town hall. An observer stands 65 feet from the point on the ground directly below the flagpole. He finds the angles of elevation of the top and bottom of the flagpole to be 49° and 45° respectively. Find the length of the flagpole.

**33.** A man standing on one bank of a stream found the angle of elevation of the top of a tree directly opposite to him on the other bank to be 24°. He then walked 100 feet away from the stream in line with the foot of the tree and his original position. The new angle of elevation was 17°. Find the width of the stream.

CHAPTER XVII

# AN INTRODUCTION TO QUADRATICS

*"I'm very well acquainted too with matters mathematical,*
*I understand equations, both the simple and quadratical."*[1]

W. S. Gilbert

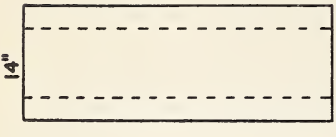
## *Re-enter the Equation*

You will recall that algebra provides a good method of olving many kinds of problems. This is particularly true for roblems whose solutions require the finding of an unknown umber or numbers. *Let the unknown number be x* is one of ie commonest statements in algebra. Our general plan of tack is to obtain from the data of the problem an equation iat involves the unknown number. Then we find the value this number by solving the equation. On page 20, we sted some basic principles by means of which any equation et in this book can be solved.

In order to get acquainted once more with some equations, t us consider a problem confronting Mr. Eves, the local ismith.

*xample.* Mr. Eves wishes to make a trough from a long rip of galvanized iron 14 inches wide. He can do this by easing the metal along two lines parallel to the edges and en bending the edges through a right angle (see page 350). To t the proper carrying capacity, the cross-section of the ough must be 24 sq. in. Where should the metal be creased?

[1] *he Pirates of Penzance* in *The Savoy Operas*, published by Macmillan Company Limited, London, England.




Let the creases be made $x$ inches from the edges. The
the depth of the trough is $x$ inches and its width, $14 - 2$
inches. This gives us the equation

$$x(14 - 2x) = 24$$

or $$14x - 2x^2 = 24,$$

whence $$x^2 - 7x + 12 = 0. \quad . \quad . \quad . \quad (1)$$

Try to solve equation (1). * * *

Did you find the equation a little stubborn? What cause
the difficulty? Wherein does (1) differ from the equations yo
have worked with before?

* * *

In the past, our attention has been confined to equatio
involving the unknown number or numbers to the first pow
only. Equation (1) involves the ***second power of x.*** Su
an equation is called a ***quadratic equation in x,*** since
contains the square of $x$ but no higher power (Lat. *quadratu*
square). How can we solve equation (1)? To answer th
question, first consider the equation

$$ab = 0 \quad . \quad . \quad . \quad ($$

What conclusion can be drawn from the fact that the produ
of the two numbers $a$ and $b$ is zero?

* * *

You see that this equation is satisfied if $a = 0$, regardle
of the value of $b$; the equation is also satisfied if $b = 0$, r
gardless of the value of $a$. Moreover there is no other way

hich the product $ab$ can be zero. Note that if $a = 0$, the alue of $b$ may or may not be 0; it doesn't matter about the alue of $b$, since the fact that $a = 0$ is enough to guarantee hat the product $ab$ equals 0. A similar statement holds if $b = 0$. hus in order to get all possible ways of satisfying the equation, e must consider ***both*** cases: (1) $a = 0$, and (2) $b = 0$.

> ***Basic Principle***
>
> *If the product of two numbers is zero, then* ***at least*** *one of the numbers must be zero; moreover, the product of two numbers is zero if* ***either one*** *of the numbers is zero.*

### EXERCISES

**1.** What conclusion can be drawn if (*a*) $xy = 0$? (*b*) $xyz =$ ? (*c*) $x(y - 3) = 0$? (*d*) $(x - 2)\,y = 0$? (*e*) $(x - 2)(x - 3) = 0$?

**2.** Solve (*a*) $(x - 7)(x - 5) = 0$; (*b*) $(x - 7)(x + 5) = 0$; ) $(x + 7)(x + 5) = 0$.

* * *

Let us return to our equation $x^2 - 7x + 12 = 0$. What ocedure is suggested by the exercises you have just done? ou see that if we can factor the left side of the equation, pressing it as the product of two linear factors, the basic inciple can be used as in Ex. 2 above. The expression

$$x^2 - 7x + 12$$

called a ***trinomial*** since it consists of three terms. It also called a ***quadratic expression*** since it contains the ***uare*** of $x$ but no higher power. Try to factor this trinomial.

A little intelligent guesswork probably brought you the following result:

$$x^2 - 7x + 12 = (x - 4)(x - 3).$$

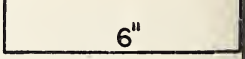
We are now in a position to answer the tinsmith's problem. Since

$$(x - 4)(x - 3) = 0,$$

the basic principle on page 351 gives us that

$$\textit{either } x - 3 = 0, \quad \textit{or} \quad x - 4 = 0.$$

Therefore $x$ must be equal to either 3 or 4. Both of these answers offer solutions to the problem. The tinsmith can crease the metal either 3 inches or 4 inches from the edges and get the proper area of cross-section (see Fig.).

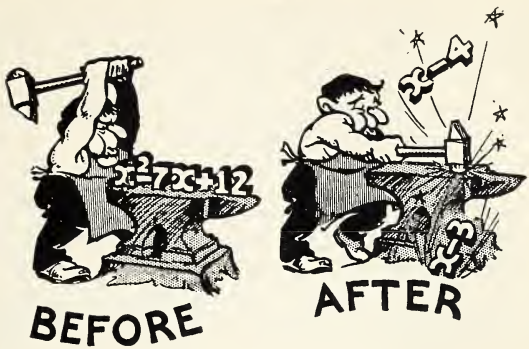



**EXERCISE**

Work out a tinsmith's problem like that in the example on page 349, given that the strip of galvanized iron is 16 inches wide and the area of cross-section must be 30 square inches.

## *We Factor a Trinomial*

The solution of the quadratic equation in the tinsmith's problem suggests that sometimes it is quite important to be able to find the factors of a trinomial. Let us gain some skill in this process.




You have likely discovered for yourself that the factors of

$$x^2 - 7x + 12$$

an be found by trial and error; that is, by shrewd guessing. t is seen that the $x^2$ is the product of $x$ and $x$; on the other hand, he last term 12 is the product of several pairs of integers, amely 12 and 1, $-12$ and $-1$, 6 and 2, $-6$ and $-2$, 4 and , $-4$ and $-3$. This suggests as possible factors the follow- ng:

$$\begin{array}{cccccc} x+12 & x-12 & x+6 & x-6 & x+4 & x-4 \\ \underline{x+1} & \underline{x-1} & \underline{x+2} & \underline{x-2} & \underline{x+3} & \underline{x-3} \end{array}$$

Notice that we are performing an operation which is just he reverse of multiplying two binomials. We are given the roduct, and we wish to find the binomial factors. The roducts of the six suggested pairs of factors above all contain he first term $x^2$ and the last term 12. Which contains the iddle term $-7x$? A little mental arithmetic shows that the nly product which contains this term is the last one. Hence

$$x^2 - 7x + 12 = (x - 4)(x - 3).$$

lere follow some examples with which to experiment.

*Examples.* Factor: (*a*) $x^2 + 5x + 6$; (*b*) $x^2 - 5x + 6$; (*c*) $x^2 + 8x + 7$; (*d*) $x^2 - 8x + 7$; (*e*) $x^2 - 2x - 8$; (*f*) $x^2 + 2x - 8$; (*g*) $x^2 + 2x - 15$; (*h*) $x^2 - 2x - 15$.

Use the trial and error method to find these factors. Then compare your answers with those below.

* * *

(*a*) $(x + 3)(x + 2)$; (*b*) $(x - 3)(x - 2)$; (*c*) $(x + 7)(x + 1)$; (*d*) $(x - 7)(x - 1)$; (*e*) $(x - 4)(x + 2)$; (*f*) $(x + 4)(x - 2)$; (*g*) $(x + 5)(x - 3)$; (*h*) $(x - 5)(x + 3)$.

Do you observe anything about these factors that would help you to gain speed? * * * For each pair of factors note the sum and product of the numerical terms, and compare them with the numerical parts of the given trinomial. For example, in the factors of (*a*), the sum of 3 and 2 is 5 and the product, 6; in the trinomial (*a*), 5 is the coefficient of $x$ and 6 is the constant term. Similarly, in the factors of (*h*) $-5$ and $+3$ have a sum $-2$ and a product $-15$; these numbers are respectively the coefficient of $x$ and the constant term in the given trinomial (*h*). In finding factors for $x^2 + 9x + 20$, we should get factors whose numerical terms have a sum of __?__ and have a product of __?__ .

> *If $x^2 + Sx + P$ is factored into two linear factors, the numerical parts of these linear factors have the sum $S$ and the product $P$.*

It is apparent that if the trinomial isn't rather carefully selected, we might find great difficulty in guessing its factors. For example, the trinomial $x^2 + 2x + 2$ requires for its factoring two numbers whose sum is 2 and whose product is also 2. You would never be able to guess such numbers. For th

ime being, therefore, our trinomials will all be of the hand-icked variety. Later, if you continue your study of algebra, ou will learn methods of dealing with more difficult types.

### EXERCISES

actor the following trinomials:

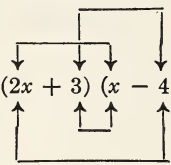
1. $x^2 + 7x + 10$.
2. $x^2 - 7x + 10$.
3. $x^2 + 3x - 10$.
4. $x^2 - 3x - 10$.
5. $a^2 + 15a + 50$.
6. $a^2 - 15a + 50$.
7. $x^2 + 4x + 3$.
8. $x^2 - 4x + 3$.
9. $x^2 - 2x - 3$.
10. $x^2 + 2x - 3$.
11. $m^2 + 12m + 36$.
12. $m^2 - 12m + 36$.
13. $x^2 - 18x + 45$.
14. $x^2 + x - 20$.
15. $a^2 - a - 20$.
16. $k^2 + 12k + 35$.
17. $x^2 - 11x + 18$.
18. $x^2 + 2x + 1$.
19. $n^2 - 3n + 2$.
20. $x^2 - 3x - 28$.
21. $a^2 + 3a - 28$.
22. $x^2 + 30x + 125$.
23. $b^2 - 33b + 90$.
24. $a^2 - a - 12$.
25. $x^2 + x - 12$.
26. $x^2 - 27x + 50$.
27. $m^2 - m - 30$.
28. $m^2 + m - 30$.
29. $x^2 + 13x + 42$.
30. $x^2 - 18x + 45$.
31. $x^2 - x - 56$.
32. $k^2 + k - 56$.
33. $x^2 + 2x - 48$.
34. $x^2 - 5x - 36$.

## *We Improve Our Skill*

The product of two binomials like $2x + 3$ and $x - 4$ esults in a trinomial: $2x^2 - 5x - 12$. Note that the coeffi-ient of $x^2$ is *not* 1, as in trinomials of the previous exercises. Vhen we factor such a trinomial by inspection, we try by orking backwards to discover the two binomials which give

rise to it. If you have facility in multiplying binomials, you will likely be able to find factors quickly. The time has come when you should be able to multiply binomials with speed and accuracy. Study the two methods below.

$$\begin{array}{r} 2x + 3 \\ x - 4 \\ \hline 2x^2 + 3x \\ \quad - 8x - 12 \\ \hline 2x^2 - 5x - 12 \end{array}$$




The horizontal method is faster. You multiply the $2x$ by $x$ to get the $2x^2$; multiply 3 by $x$ and $2x$ by $-4$ and add the results mentally, getting $-5x$; finally, multiply 3 by $-4$ getting $-12$. The arrows indicate the procedure. Try this method in the following exercises.

## EXERCISES

Multiply the following factors at sight:

1. $(x + 7)(x + 4)$.
2. $(x - 7)(x - 4)$.
3. $(x - 7)(x + 4)$.
4. $(x + 7)(x - 4)$.
5. $(2x + 1)(x + 3)$.
6. $(3x + 2)(x + 5)$.
7. $(2x - 1)(x + 3)$.
8. $(3x - 2)(x + 5)$.
9. $(4x + 3)(x + 3)$.
10. $(4x + 3)(x - 5)$.
11. $(5x - 4)(2x + 1)$.
12. $(5x - 4)(2x - 1)$.
13. $(3x + 2)(2x + 3)$.
14. $(3x - 2)(2x - 3)$.
15. $(x - 5)(3x + 5)$.
16. $(x - 5)(3x - 5)$.
17. $(2x - 6)(2x - 6)$.
18. $(5x - 2)(3x + 4)$.
19. $(7x + 1)(3x + 1)$.
20. $(7x - 1)(3x + 1)$.
21. $(7x - 1)(3x - 1)$.
22. $(7x + 1)(3x - 1)$.
23. $(5y + 2)(2y - 3)$.
24. $(3a + 1)(5a - 4)$.
25. $(9b + 5)(b - 2)$.
26. $(8x - 3)(2x + 1)$.
27. $(2m + 5)(2m - 5)$.
28. $(6k - 5)(2k + 3)$.
29. $(5x - 8)(3x + 4)$.
30. $(7x + 4)(5x - 4)$.

With the method of multiplying binomials clearly in mind, you will be able readily to construct factors of trinomials such as those in the following examples. (Note that in these examples the coefficient of the term with $x^2$ is not 1.)

*Examples.* Factor (*a*) $2x^2 + x - 6$, (*b*) $3x^2 + 11x - 20$.

(*a*) Since $2x^2$ is the product of $2x$ and $x$, and 6 is the product of 6 and 1 or 3 and 2, we have for possible factors
$(2x \quad 6)(x \quad 1)$, $(2x \quad 1)(x \quad 6)$, $(2x \quad 3)(x \quad 2)$, $(2x \quad 2)(x \quad 3)$, in which the numerical terms in each pair have opposite signs. Examine the possible terms in $x$ by the method suggested by the arrows on page 356. The third set is the only one that can yield a term $+ x$. Therefore, the factors are $(2x - 3)(x + 2)$.

(*b*) Possible factors are the following, in which again the numerical terms have opposite signs:
$(3x \quad 20)(x \quad 1)$, $(3x \quad 1)(x \quad 20)$, $(3x \quad 10)(x \quad 2)$,
$(3x \quad 2)(x \quad 10)$, $(3x \quad 5)(x \quad 4)$, $(3x \quad 4)(x \quad 5)$.

The only pair of factors that can give a term $+ 11x$ is

$$(3x \quad 4)(x \quad 5).$$

Attention to the signs shows that the required factors are

$$(3x - 4)(x + 5).$$

After a little experience, you will be able quickly to rule out the inadmissible pairs of factors by inspection, so that you will not need to write down all the possible pairs.

### EXERCISES

Factor the following trinomials. Check your factors mentally by multiplication.

1. $2x^2 + 5x + 3$.
2. $2x^2 + 5x - 3$.
3. $2x^2 + 7x + 5$.
4. $2x^2 - 3x - 5$.

5. $3x^2 + 11x + 6$.
6. $3x^2 + 7x - 6$.
7. $3x^2 - 11x + 6$.
8. $3x^2 - 7x - 6$.
9. $2x^2 - x - 6$.
10. $2x^2 + 7x + 6$.
11. $2x^2 + 14x + 20$.
12. $2x^2 + 3x - 20$.
13. $3x^2 - 13x + 12$.
14. $3x^2 + 17x - 28$.
15. $5x^2 - 8x + 3$.
16. $5x^2 + 2x - 3$.
17. $4x^2 + 8x + 3$.
18. $4x^2 - 8x + 3$.
19. $4x^2 - 4x - 3$.
20. $4x^2 - 7x + 3$.
21. $6x^2 + 13x - 5$.
22. $6x^2 + 31x + 5$.
23. $7x^2 - 18x + 8$.
24. $7x^2 - 15x + 8$.
25. $2x^2 - 3x - 9$.
26. $2x^2 + 7x - 9$.
27. $10x^2 + 19x + 7$.
28. $10x^2 - 17x + 7$.
29. $3x^2 - 4x - 15$.
30. $3x^2 + 12x - 15$.
31. $13x^2 - 11x - 2$.
32. $12x^2 + 15x + 3$.
33. $12x^2 + 16x - 3$.
34. $12x^2 - 35x - 3$.
35. $5x^2 + 22x - 15$.
36. $5x^2 - 28x + 15$.
37. $5x^2 + 74x - 15$.
38. $8x^2 + 10x - 7$.
39. $8x^2 - x - 7$.
40. $2x^2 + 13x - 24$.

## *Back to the Quadratic Equation*

The tinsmith's problem at the beginning of this chapter led us to a quadratic equation. In attempting to solve the quadratic equation, we found it necessary to be able to factor a trinomial. Did you solve the second tinsmith's problem on page 352? If the metal sheet is creased $x$ inches from the edge, the problem leads to the equation

$$x\,(16 - 2x) = 30.$$

Whence $16x - 2x^2 = 30$, or $x^2 - 8x + 15 = 0$.

Therefore, we have

$$(x - 5)\,(x - 3) = 0,$$

which gives $x = 5$, or $x = 3$.

Once again both values of $x$ provide solutions to the prob-
em. The proper area of cross-section can be obtained by
reasing the metal either 3 inches or 5 inches from the edge.

This example suggests the following method of solution for
ertain quadratic equations.

> ***If the right side of a quadratic equation is zero, then the equation can be solved if the left side can be factored into two linear factors.***

*Example* 1. Solve $x^2 - x - 20 = 0$.
We have $(x - 5)(x + 4) = 0$.
Therefore $x - 5 = 0$, or $x + 4 = 0$.
This gives $x = 5$ or $-4$.
Verification: If $x = 5$, left side $= 25 - 5 - 20 = 0$.
If $x = -4$, left side $= 16 - (-4) - 20 = 0$.

*Example* 2. Find two consecutive whole numbers whose
oduct is 42.

Let $x$ and $x + 1$ be the numbers. Then we have
$x(x + 1) = 42$,
or $x^2 + x - 42 = 0$.
Therefore $(x + 7)(x - 6) = 0$,
whence $x + 7 = 0$, or $x - 6 = 0$.
This gives us that $x = 6$, or $x = -7$.

It is important to note that in this case both values of $x$
ve answers to the problem. If $x = 6$, the numbers are 6
id 7. If $x = -7$, the numbers are $-7$ and $-6$.

Verify these answers by seeing that they satisfy the con-
tions of the problem. You could probably have guessed an
nswer to this problem without using any algebra. However,

the algebraic method is interesting because it shows that the problem has two and only two solutions.

**Some facts about quadratic equations.** The essential term in a quadratic equation is that which contains $x^2$; either one or both of the other terms may be missing. Thus, the following are both quadratic equations:

$$x^2 = 25, \quad . \ . \ . \quad (1)$$

$$x^2 - 4x = 0. \quad . \ . \ . \quad (2)$$

There is no difficulty in solving these special types. You see at once that in (1) we have $x = +5$ or $-5$. The terms on the left side of (2) have a common factor, so that

$$x(x - 4) = 0,$$

whence $x = 0$, or $x - 4 = 0$. Thus $x = 0$ or 4.

Both numbers 0 and 4 satisfy equation (2). *Numbers that satisfy an equation are called* ***roots*** *of the equation.* Thus, 0 and 4 are roots of (2) and $+5$ and $-5$ are roots of (1). Since all quadratics considered so far have had ***two*** roots, you will not be surprised to learn that ***every quadratic equation has two roots.*** Sometimes students conclude that quadratic equations like (2) have only one root.

They argue thus:

Divide both sides of (2) by $x$, and get

$$x - 4 = 0, \text{ whence } x = 4.$$

What is wrong with this method?

* * *

The error lies in the failure to recall a very important principle:

***Division by zero is not allowed in mathematics.***

There is no way of dividing by zero so as to get useful results, so mathematicians rule out division by zero as an operation. For example, we have

$$0 \times 34 = 0 \times 98.$$

If we allow division by zero, we get the absurd result $34 = 98$!

In the example quoted above, the division by $x$ is valid only if $x \neq 0$.* Thus, if $x \neq 0$, then $x = 4$. However, the basic principle on page 351 tells us that in order to get all possible solutions of $x(x - 4) = 0$, we must consider the possibility that the first factor is zero as well as the possibility that the second factor is zero. It is precisely the first of these two possibilities that has been overlooked in the incorrect solution above. If now the case in which $x = 0$ is discussed, we find by substitution that 0 is another root of (2). Here is a suggestion worth noting: *In solving an equation, do not divide out any factor which involves the unknown.*

It is also well to observe at this point that algebra is sometimes over-generous in providing answers, occasionally giving more than is asked. For example, in a given problem leading to a quadratic equation, the two roots of the equation may not both give answers to the problem. Thus, in Example 2 above, if the problem had read *Find two consecutive* ***negative*** *whole numbers* etc., then the root $x = 6$ would have failed to furnish an answer to the problem. The conclusion is that $x$ is 6 ***or*** $-7$ in the ***equation;*** answers to the problem, if any, are found among the roots of the ***equation.***

### EXERCISES

Solve the following quadratic equations. Verify your answers:

1. $x^2 - 5x + 6 = 0$.
2. $x^2 + 7x + 12 = 0$.
3. $x^2 - x - 12 = 0$.
4. $x^2 - 7x + 10 = 0$.
5. $x^2 - 11x + 28 = 0$.
6. $x^2 + 15x + 50 = 0$.

* $a \neq b$ means that $a$ is not equal to $b$.

7. $x^2 - 3x + 2 = 0$.
8. $x^2 - x - 56 = 0$.
9. $x^2 - 7x = 0$.
10. $x^2 + 5x = 0$.
11. $x^2 = 36$.
12. $x^2 - 49 = 0$.
13. $2x^2 + 7x + 3 = 0$.
14. $3x^2 + 17x + 10 = 0$.
15. $2x^2 + 5x - 3 = 0$.
16. $3x^2 - 12 = 0$.
17. $3x^2 - 12x = 0$.
18. $5x^2 + 12x + 4 = 0$.
19. $12x^2 + 5x - 3 = 0$.
20. $3x^2 - 20x + 25 = 0$.
21. $x^2 - 28x + 75 = 0$.
22. $2x^2 + 3x - 9 = 0$.
23. $8x^2 + x - 7 = 0$.
24. $x^2 - 21x - 100 = 0$.

Solve each of the following problems by introducing one unknown and solving the quadratic equation that arises.

25. Find two consecutive whole numbers whose product is 72.

26. Find two consecutive negative whole numbers whose product is 56.

27. Find two consecutive even whole numbers whose product is 80.

28. A farmer has 100 rods of fence with which he wishes to enclose a rectangular field with area 600 square rods. Find the dimensions of the field.

29. The perimeter of a rectangular pane of glass is 44 inches and its area is 117 square inches. Find the dimensions of the pane.

30. Find two consecutive whole numbers such that the sum of their squares is 85.

31. If three times a certain number is added to twice its square, the sum is 9. Find the number.

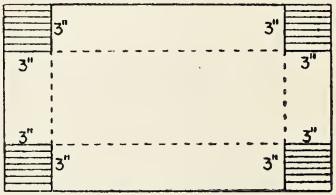
32. The numbers of inches in the three sides of a right-angled triangle are consecutive whole numbers. Find the lengths of the sides.

33. The length of a rectangular sheet of paper exceeds the width by 4 inches and its area is 96 square inches. Find the dimensions.

34. The base of a triangle is 3 inches less than its height, and the area is 65 square inches. Find the base and height.

35. A squad of 106 soldiers is arranged in a formation consisting of two squares. If there are 4 more soldiers in the front row of the large square than in the front row of the small one, find the number of soldiers in each square.

36. A rectangular piece of cardboard is 5 inches longer than it is wide. A box without a lid is made from it by cutting a 3-inch square from each corner of the rectangle (see Fig.), folding up the four flaps of the cardboard remaining, and gluing the edges. If the capacity of the box is 108 cubic inches, find the original dimensions of the cardboard.



CHAPTER XVIII

# MORE ABOUT ALGEBRA

*"Let us grasp the situation,*
*Solve the complicated plot—*
*Quiet, calm deliberation*
*Disentangles every knot."*[1]

W. S. Gilbert

## Crossroads

Most of the work in the sections following is primarily fo those who are planning to proceed to further work in mathe matics. Its purpose is to help such students gain ease an facility in dealing with algebraic operations. This extensio of your present skill is necessary in order that too great burden is not left for your next course.

Many of the manipulations will be developed from ideas i arithmetic with which you should be familiar. To thi extent, the work will possibly throw some additional light o the processes of arithmetic. However, other parts of thi course will be of more immediate use to you if this is to b your last year of school mathematics.

For those who plan to continue their mathematical studie the joint objectives of speed and accuracy in working with th symbols of algebra are very important. You will find tha many of the exercises which follow are just extensions o skills and processes already met in Book I.

[1]From *The Gondoliers* in *The Savoy Operas*, published by Macmillan Company Limited, London, England.

## *Multiplication*

Certain types of product are required so frequently that you should memorize them as formulas. Here they are:

$$
\begin{aligned}
(a + b)^2 &= a^2 + b^2 + 2ab \\
(a - b)^2 &= a^2 + b^2 - 2ab \\
(a + b + c)^2 &= a^2 + b^2 + c^2 + 2ab + 2bc + 2ca \\
(a + b)\,(a - b) &= a^2 - b^2 \\
(a + b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3 \\
(a - b)^3 &= a^3 - 3a^2b + 3ab^2 - b^3
\end{aligned}
$$

Verify these formulas by multiplication, then learn them. It will add to your understanding of the formulas if you practise translating them into statements. Thus the first one may be stated: *The square of the sum of two numbers equals the square of the first number plus the square of the second plus twice the product of the two numbers.* Likewise the fourth formula in the list gives the rule: *The sum of two numbers times their difference equals the square of the first number less the square of the second.* After you have learned the formulas, test your ability on the following exercises.

### EXERCISES

*(Answer 1-33 orally)*

State the results of the indicated products:

1. $(x + y)^2$.
2. $(m - n)^2$.
3. $(x + y + z)^2$.
4. $(x + y)\,(x - y)$.
5. $(x + y)^3$.
6. $(x - y)^3$.
7. $(2x - y)^2$.
8. $(3m + 2n)^2$.
9. $(a + 8)\,(a - 8)$.
10. $(2a + b)^3$.
11. $(a + b - c)^2$.
12. $(3a + 7)^2$.

**13.** $(a + 4)^3$.
**14.** $(2x - y)(2x + y)$.
**15.** $(4b - 3)^2$.
**16.** $(5p - 4q)^2$.
**17.** $(x - 5)^3$.
**18.** $(3x + 5y)^2$.
**19.** $(a + b + 2c)^2$.
**20.** $(a + b - 2c)^2$.
**21.** $(3a - 5)(3a + 5)$.
**22.** $(50 + 1)(50 - 1)$.
**23.** $(x + 2y)^3$.
**24.** $(x - 2y)^3$.
**25.** $(7a - 6)^2$.
**26.** $(7x + 3y)^2$.
**27.** $(2k - 5m)^2$.
**28.** $(a - b - c)^2$.
**29.** $(5k - 2m)(5k + 2m)$.
**30.** $(9x - 3)(9x + 3)$.
**31.** $(8x - 3y)^2$.
**32.** $(x + 2y - z)^2$.
**33.** $(4a + 3b)^2$.

**34.** Find the area of a rectangle whose sides are (*a*) $x - 2y$ and $x + 2y$, (*b*) $x + y$ and $x^2 - xy + y^2$, (*c*) $5a + 3b$ and $3a - 5b$.

**35.** Find the area of a square whose side is (*a*) $4x - y$, (*b*) $5x + 4y$, (*c*) $x + y/2z$.

**36.** Find the volume of a cube whose edge is $x - 2y$ inches.

**37.** Find the area of a triangle whose base is $a + 3b$ feet and whose altitude is $2a - b$ feet.

**38.** A pyramid has a square base with side $3x + 2y$ inches and an altitude of $6x$ inches. Find the volume of the pyramid.

**39.** A circular cylinder is $3a + b$ feet high and the radius of its base is $3a - b$ feet. Find its volume.

**40.** The slant height of a cone is $5m - n$ inches and the radius of its base is $3m + n$ inches. Find the area of the lateral surface of the cone.

* * *

The use of these standard products in simplification is illustrated in the following example.

*Example* 1. Simplify $(a + 1)^2 - (a + 1)(a - 1) - 2(a + 1)^3$

The expression is equal to

$$\begin{aligned} &a^2 + 1 + 2a - (a^2 - 1) - 2(a^3 + 3a^2 + 3a + 1) \\ &= a^2 + 1 + 2a - a^2 + 1 - 2a^3 - 6a^2 - 6a - 2 \\ &= -2a^3 - 6a^2 - 4a. \end{aligned}$$

Recall that ***parentheses mean multiplication.*** The factor $-2$ before the third term means that this term is to be ***multiplied*** by $-2$. As a matter of procedure, *it is best not to try two multiplications at once* until you become skilful. For example, in the solution to Example 1, the product of $a + 1$ and $a - 1$ is found first, then this product is multiplied by $-1$ in the next step.

We have multiplied binomials together before. The multiplication of trinomials and longer expressions offers little that is new. You should be able to write the work down in either a vertical or a horizontal arrangement as below.

*Example* 2.

$$\begin{array}{r} 3x^2 - 2xy + 5y^2 \\ \underline{2x^2 + 5xy - 4y^2} \\ 6x^4 - 4x^3y + 10x^2y^2 \\ \quad 15x^3y - 10x^2y^2 + 25xy^3 \\ \quad\quad - 12x^2y^2 + 8xy^3 - 20y^4 \\ \hline 6x^4 + 11x^3y - 12x^2y^2 + 33xy^3 - 20y^4 \end{array}$$

If $x = 1$, $y = 2$.

$$\begin{array}{rl} 3 - 4 + 20 = & 19 \\ 2 + 10 - 16 = & \underline{-4} \\ & -76 \end{array}$$

$$6 + 22 - 48 + 264 - 320 = -76$$

Written horizontally, the work becomes

$$\begin{aligned} &(3x^2 - 2xy + 5y^2)(2x^2 + 5xy - 4y^2) \\ &\quad = 6x^4 + 15x^3y - 12x^2y^2 - 4x^3y - 10x^2y^2 + 8xy^3 \\ &\qquad + 10x^2y^2 + 25xy^3 - 20y^4 \\ &\quad = 6x^4 - 11x^3y - 12x^2y^2 + 33xy^3 - 20y^4. \end{aligned}$$

It is important to keep the work orderly. By letting $x$ and $y$ take particular values, we can obtain a useful check.

### EXERCISES

Simplify the expressions on the following page by performing the indicated multiplications and collecting like terms.

1. $(x + y)^2 - (x + y)(x - y) + 2(x - y)^2$.
2. $(a + b + c)^2 - (a + b)^2 - (b - c)^2$.
3. $(x + 1)(x - 1) - 2(x - 1)^2 + (x + 1)^2$.
4. $(a + b)^3 - (a - b)^3$.
5. $(3x - 4y)^2 - 3(2x + y)(2x - y) - (2x + 3y)^2$.
6. $-(m - 3n)^2 - 2(5m + 2n)(2m - n) + (5m + 4n)^2$.
7. $(x - 5)(2x^2 - 3x + 7) - (2x + 3)(x^2 + 3x - 12)$.
8. $(5a^2 - 2ab + b^2)(3a^2 + ab - 4b^2)$.
9. $(2x - y - z)^2 - (x + y + z)^2$.
10. $(3k + m)(5k - 6m) - 5(k - 3m)(3k + m) + (2k + 5m)(2k - 5m)$.
11. $(3x - 2)(2x^2 - 5x + 6) - (x + 4)(5x^2 - x - 2) + (x + 2)^3$.
12. $4(2x + 5)^2 + 5(x - 3)^2 - 13(x + 5)(x - 5)$.
13. $(x - 1)(2x + 3)(x + 5) - (2x - 5)(x + 3)(x - 3)$
14. $(x^2 - 5x + 8)(4x^2 + 3x + 3) - 2(2x^2 + 5)(x^2 + 6x - 5)$.
15. $(a + 2b - c)^2 + (2a - b)^2 - (b - 2c)^2 + (a + c)^2$.
16. $(3x - 2y)(3x + 2y) + (4x + y)(4x - y) - (5x + 3y)(5x - 3y)$.
17. $(2p + q)^3 + (2p - q)^3$.
18. $(x - 2)(x^2 + 3x - 4) + (x + 3)(2x^2 - x - 1) - 3(x - 1)(x^2 + 1)$.
19. $(k + m - n)^2 + (k - m + n)^2 - 2(k - m - n)^2$.
20. $5(3x + 4)^2 - 3(5x - 7)^2 + 30(x + 10)(x - 10)$.

## *Division*

We have learned how to divide an expression by a monomial. Recall the laws of division, and satisfy yourself that the following example is correct.

*Example* 1.

$$\frac{25x^3y^2 - 10x^2y^3 + 15xy^4}{-5xy^2} = -5x^2 + 2xy - 3y^2.$$

The extension to division with more terms in the divisor is similar to long division in arithmetic. Compare the two processes in the next example.

*Example* 2.

$$\begin{array}{r|l} 347)\ 13568\ (39 & x^2-5x+7)\ x^4+9x^3-4x^2-7x+12\ (x^2+14x+59 \\ \underline{1041} & \underline{x^4-5x^3+7x^2} \\ 3158 & 14x^3-11x^2-7x \\ \underline{3123} & \underline{14x^3-70x^2+98x} \\ 35 & 59x^2-105x+12 \\ & \underline{59x^2-295x+413} \\ & 190x-401 \end{array}$$

*Example* 3. Divide $6x^3 - 7x^2 - 50$ by $2x - 5$.

Try this yourself first, then compare your result with that below.

* * *

$$\begin{array}{r|l} x-5)\ 6x^3-7x^2+0-50\ (3x^2+4x+10 & \underline{\text{Check: Let } x = 1.} \\ \underline{6x^3-15x^2} & -3)6-7-50(3+4+10 \\ 8x^2+0 & \\ \underline{8x^2-20x} & \\ 20x-50 & \text{or} \quad -3)-51(17 \\ \underline{20x-50} & \\ 0 & \end{array}$$

At first you may find it useful to put in zeros for missing terms as shown. This helps to keep the work in order. You will find it best to see that the divisor and dividend are both arranged so that the indices of the powers either always increase or always decrease as you go from left to right. Thus the expression $7 + x - 2x^3 + 3x^2$ should be arranged either as $7 + x + 3x^2 - 2x^3$ (*ascending powers of* $x$), or as $2x^3 + 3x^2 + x + 7$ (*descending powers of* $x$).

### EXERCISES

Divide:

1. $16x^3y^2 - 8x^2y^3 + 24xy^4$ by $-8xy^2$.

2. $4a^2b^2 + 6ab^3 - 2a^3b^2$ by $2ab^2$.

3. $x^2 - 5x + 6$ by $x - 3$. Check your answer.

4. $a^2 - 2ab - 15b^2$ by $a + 3b$. Check your answer.

5. $x^3 + 2x^2 + x + 2$ by $x + 2$. Check your answer.

6. $8x^3 + 8x^2y + 4xy^2 + y^3$ by $2x + y$. Check you answer.

7. $3x^3 - 5x^2 + 3x - 1$ by $x - 1$.

8. $x^2 - 16y^2$ by $x - 4y$.

9. $x^3 - 5x + 4$ by $x - 1$.

10. $a^3 + a - 10$ by $a - 2$.

11. $21n^3 + 10n^2 - 31n + 10$ by $3n - 2$. Check you answer.

12. $3x^4 - 5x^3y - 12x^2y^2 - xy^3 + 3y^4$ by $3x^2 + xy - y^2$ Check when $x = 1$, $y = 2$.

13. $a^4 + 2a^3 + 3a^2 + 2a + 1$ by $a^2 + a + 1$. Check when $a = 2$.

14. $12x^3 - 19x^2 + 13x - 12$ by $3x - 4$. Check whe $x = 1$.

15. $a^3 + 3a^2b + 3ab^2 + b^3$ by $a^2 + 2ab + b^2$. Chec when $a = 1$, $b = 2$.

16. $m^3 - 8m - 3$ by $3 - m$.

17. $2a^3 - 5a^2 + 4$ by $-a + 2$.

18. $z^4 + 2z^3 + z + 2$ by $1 - z + z^2$. Check when $z =$

19. $x^4 + x^2y^2 + y^4$ by $x^2 - xy + y^2$.

20. $a^3 + b^3$ by $a + b$.

**21.** The area of a rectangle is $9a^2 - 25$ and its length is $a + 5$. Find the width.

**22.** The area of a rectangle is $m^4 + m^2n^2 + n^4$ sq. ft. and its width is $m^2 + mn + n^2$ feet. Find its length.

**23.** If the area of a triangle is $x^3 - y^3$ sq. cm. and its altitude, $x - y$ cm., find the length of its base.

## *Factoring*

In the course of our work so far, we have encountered three kinds of factoring.

| *Type* | *Example* |
|---|---|
| *Common factor* | $ax^2 - a^2x = ax(x - a)$ |
| *Difference of squares* | $x^2 - y^2 = (x - y)(x + y)$ |
| *Trinomial* | $x^2 - 15x + 56 = (x - 7)(x - 8)$ |

Each of these types has proved useful. Taking out a common factor sometimes simplifies a formula: for example, the area of a circular ring is given by

$$\pi R^2 - \pi r^2 \text{ or } \pi(R^2 - r^2).$$

This same formula is further improved for the purposes of computation by factoring the difference of the two squares, thus:

$$\pi(R + r)(R - r).$$

Moreover, the factoring of a trinomial enables us to solve certain quadratic equations. Later on, we shall see that other types of equations can be solved by factoring. In addition to these uses, the factored form of an expression is often the easier form in which to substitute numbers, as well

as being a less complex form for many operations. As in arithmetic, we shall find that factoring simplifies some operations with fractions.

*Examples.* Factor: (*a*) $m^2 - 9$; (*b*) $2x^2 - x - 1$; (*c*) $a^4 - a^2$; (*d*) $ax^2 - 5ax + 4a$.

Try these first for yourself.

* * *

Observe that (*c*) and (*d*) involve two kinds of factoring. It is a good plan to ***take out a common factor, if any, first.***

We have

(*a*) $m^2 - 9 = (m + 3)(m - 3)$,

(*b*) $2x^2 - x - 1 = (2x + 1)(x - 1)$,

(*c*) $a^4 - a^2 = a^2(a^2 - 1) = a^2(a + 1)(a - 1)$,

(*d*) $ax^2 - 5ax + 4a = a(x^2 - 5x + 4) = a(x - 4)(x - 1)$.

### EXERCISES

Express the following in the simplest factored form.

1. $4x - 8y$.
2. $ax^2 + ay^2$.
3. $x^2 - 8x + 15$.
4. $m^2 - n^2$.
5. $3x^2 - 6x$.
6. $a^2 - 9$.
7. $2x^2 + x - 1$.
8. $2x^2 - 8$.
9. $10x^2y^2 + 15xy$.
10. $4x^2 - 1$.
11. $1 - 9m^2$.
12. $x^2 - 3x - 40$.
13. $z^3 - z$.
14. $2b^3 - 4b^2 + 6b$.
15. $ax^2 + 5ax + 6a$.
16. $x^4 - x^2$.
17. $3y^4 - 12y^2$.
18. $mx^3 - mx^2$.
19. $5a^2 - 20$.
20. $4x^2 - 32x + 60$.
21. $36ab - 12a^2$.
22. $2x^3 - 4x^2 - 6x$.

23. $3m^2 - 2m - 1$.
24. $x^2 - \frac{1}{4}$.
25. $n^2 - 13n + 36$.
26. $9x^2y^2 - 1$.
27. $10x^2 + 15x - 10$.
28. $1 - k^2$.
29. $1 - 4k^2m^2$.
30. $ab^3 + ab^2$.
31. $6x^3y + 12x^2y^2 - 18xy$.
32. $x^4 - 1$.
33. $1 - a^4$.
34. $x^3y + 2x^2y + xy$.
35. $2a^2 - 22a - 52$.
36. $6x^2 + 5x - 4$.
37. $km^2 - 25k$.
38. $6x^2 - 3x - 30$.
39. $a^4 - b^4$.
40. $(a + b)^2 - c^2$.
41. $(x - y)^2 - z^2$.
42. $(2c + d)^2 - 4a^2$.

43. (*a*) Factor: $x^2 - 10x + 25$.

(*b*) If the area of a square is $x^2 - 10x + 25$ sq. in., find the length of one of its sides.

44. Find the length of the side of a square whose area in uare feet is (*a*) $m^2 + 18m + 81$; (*b*) $4y^2 - 4y + 1$; (*c*) $- 12b + 36$.

* * *

The process of factoring is commonly used in algebra. In der to deal effectively with certain types of problems, you ould be able to factor expressions that admit of simple ctors. We proceed to consider some new methods of fac- ring.

## *Incomplete Squares*

Consider the following expansions of complete squares:

$$(2a + b)^2 = 4a^2 + 4ab + b^2,$$
$$(m - 5k)^2 = m^2 - 10km + 25k^2,$$
$$(x^2 - 3y^2)^2 = x^4 - 6x^2y^2 + 9y^4,$$
$$(3c^2 + 5d^2)^2 = 9c^4 + 30c^2d^2 + 25d^4.$$

By inspecting these expansions and others like them, we ar led to infer that the square of a binomial contains

(1) two terms that are perfect squares, and

(2) the proper product term.

An expression like

$$x^4 - 5x^2 + 4 \quad . \quad . \quad . \quad (1$$

suggests the perfect squares

$$(x^2 + 2)^2 \text{ and } (x^2 - 2)^2$$

except for the product term $-5x^2$. If the product term wer $-4x^2$ or $4x^2$, then (1) would be a complete square.

To form the square $(x^2 - 2)^2$, we can write (1) thus:

$$x^4 - 4x^2 + 4 - x^2 \quad . \quad . \quad . \quad (2$$

Expression (2) is equivalent to (1) and its first three term can be grouped to give the required square. Moreover, if w proceed with this grouping, we see that (1) takes the form o the difference of two squares and can be readily factored. W have

$$\begin{aligned}(x^4 - 4x^2 + 4) - x^2 &= (x^2 - 2)^2 - x^2 \\ &= (x^2 - 2 + x)(x^2 - 2 - x) \\ &= (x + 2)(x - 1)(x - 2)(x + 1\end{aligned}$$

Similarly, if we construct the square $(x^2 + 2)^2$, (1) become

$$\begin{aligned}(x^4 + 4x^2 + 4) - 9x^2 &= (x^2 + 2)^2 - (3x)^2 \\ &= (x^2 + 2 + 3x)(x^2 + 2 - 3x) \\ &= (x+1)(x+2)(x-1)(x-2).\end{aligned}$$

In each case, the completion of the square and the prope adjustment to restore the product term produced the differenc of two squares and provided a method of factoring the incom plete square. In examples such as the foregoing, it is alway possible to complete the square in two different ways However, sometimes one of these methods fails to lead t factors. For example,

$$4a^4 + 1 = 4a^4 - 4a^2 + 1 + 4a^2$$
$$= (2a^2 - 1)^2 + 4a^2,$$

hich is the *sum* of squares. Try to factor the expression by nstructing the square $(2a^2 + 1)^2$.

OTE: (1) Expressions which admit this type of factoring involve the fourth power or higher.

(2) The product term which completes the square should always be corrected by *subtracting* a like term. If the correction requires addition, you should complete the other possible square.

(3) Sometimes expressions of this type can also be factored as trinomials. For example,

$$x^4 - 5x^2 + 4 = (x^2 - 1)(x^2 - 4)$$
$$= (x - 1)(x + 1)(x - 2)(x + 2).$$

*xample* 1. $x^4 + x^2y^2 + y^4 = x^4 + 2x^2y^2 + y^4 - x^2y^2$
$= (x^2 + y^2)^2 - (xy)^2$
$= (x^2 + y^2 + xy)(x^2 + y^2 - xy).$

*xample* 2.
$$16m^4 - 44m^2n^2 + 25n^4 = 16m^4 - 40m^2n^2 + 25n^4 - 4m^2n^2$$
$$= (4m^2 - 5n^2)^2 - (2mn)^2$$
$$= (4m^2 - 5n^2 + 2mn)(4m^2 - 5n^2 - 2mn).$$

## EXERCISES

actor:

1. $m^4 + m^2n^2 + n^4$.
2. $a^4 - 5a^2 + 4$.
3. $x^4 + 3x^2 + 4$.
4. $y^4 - 10y^2 + 9$.
5. $4m^4 + 1$.
6. $4c^4 + 3c^2 + 1$.
7. $a^4 + b^4 - 18a^2b^2$.
8. $a^4 + b^4 - 14a^2b^2$.
9. $100p^4 + 25$.
10. $k^4 - 3k^2 + 1$.
11. $16x^4 + 36x^2y^2 + 25y^4$.
12. $m^8 + m^4 + 1$.
13. $25x^4 - 89x^2y^2 + 64y^4$.
14. $25x^4 + 44x^2y^2 + 64y^4$.

## *The Sum and the Difference of Cubes*

If we divide the expression $x^3 - y^3$ by $x - y$, we get th

quotient $x^2 + xy + y^2$ as shown. Similarly, if we divide $x^3 + y^3$ by $x + y$, we get the quotient $x^2 - xy + y^2$.

$$
\begin{array}{r|l}
 & x^2+xy+y^2 \\
\hline
x-y & x^3-y^3 \\
 & x^3-x^2y \\
\hline
 & x^2y-y^3 \\
 & x^2y-xy^2 \\
\hline
 & xy^2-y^3 \\
 & xy^2-y^3 \\
\hline
 & 0
\end{array}
$$

These results should be memorized:

$$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$$
$$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$$

By means of these formulas, we can factor expressions of th forms

$$(\quad)^3 \pm (\quad)^3.$$

It is seen that *the sum of the cubes of two numbers is divisible b their sum, and the difference of the cubes of two numbers i divisible by their difference.* The middle term in the secon factor of each formula should be noted carefully:

(1) the coefficient of $xy$ is *not* $\pm 2$, as in the expansions o $(a \pm b)^2$.

(2) the sign preceding $xy$ is *opposite* to that between th given cubes.

*Example* 1. $a^3 - 27 = a^3 - 3^3$
$= (a - 3)(a^2 + 3a + 9).$

*Example* 2. $8x^3 - 125y^3 = (2x)^3 - (5y)^3$
$= (2x - 5y)(4x^2 + 10xy + 25y^2).$

### EXERCISES

*For Oral Answers*

1. Translate each of the foregoing formulas into a rule English.

2. Give one factor of each of the following: (*a*) $a^3 - b^3$, $m^3 - 8$, (*c*) $125 + c^3$, (*d*) $k^3 + 27$, (*e*) $r^3 - 64$, (*f*) $8b^3 + 1$.

3. Give two factors of (*a*) $a^3 + b^3$, (*b*) $m^3 + 8$, (*c*) $x^3 - 8$, ) $1 - k^3$, (*e*) $m^3 - 1$.

*For Written Answers*

ctor:

4. $1 - 64c^3$.
5. $27x^3 + 1$.
6. $p^3 + 8q^3$.
7. $8a^3 - b^3$.
8. $x^3 + 64y^3$.
9. $27m^3 - 8$.
10. $125m^3 - 8n^3$.
11. $1 + 8c^3d^3$.
12. $216p^3 - 1000q^3$.
13. $1000a^3b^3 + 1$.
14. $2m^3 + 2n^3$.
15. $1000 - a^3b^3$.
16. $k^6 - m^3$.
17. $a^6 + b^6$.
18. $2x^3 + 54$.
19. $3k^3 - 24$.
20. $d^2 + d^5$.
21. $3x^4 - 192xy^3$.

## *Factors Found by Division*

An expression like $x^3 - 2x^2 - 5x + 6$ is sometimes called ***polynomial.*** How can we find factors of such polynomials? e method is to try to guess a factor, and then test by long vision. If our guess is indeed a factor, it will give a zero nainder when divided into the polynomial.

### EXERCISE

Use long division to find which, if any, of $x - 1$, $x - 2$, d $x - 3$, are factors of $x^3 - 2x^2 - 5x + 6$.

It is shown by long division that $x - 1$ and $x - 3$ a factors and $x - 2$ is not.

We have

$$x^3-2x^2-5x+6=(x-1)\ (x^2-x-6)=(x-1)\ (x+2)\ (x-3),$$
$$x^3-2x^2-5x+6=(x-3)\ (x^2+x-2)=(x-3)\ (x+2)\ (x-1).$$

The success of this method depends on your ability to mak shrewd guesses for factors. You see that the use of long div sion to test for factors is likely to become laborious. Th following example suggests a quicker way of testing for factor.

Since $x^2 - x - 2 = (x - 2)\ (x + 1)$,

the expression $x^2 - x - 2$ becomes zero if either $x + 1 = 0$ $x - 2 = 0$. Hence, if we make $x + 1 = 0$ by substitutin $-1$ for $x$, we get

$$x^2 - x - 2 = (-1)^2 - (-1) - 2 = 1 + 1 - 2 = 0.$$

Similarly, if we set $x = 2$, we make the factor $x - 2 =$ and the expression

$$x^2 - x - 2 = 4 - 2 - 2 = 0.$$

We may infer from the foregoing that if a polynomial equals when we set $x = 1$, then one of its factors must have become This means that $x - 1$ is a factor of the polynomial. In th same way, if the polynomial becomes 0 when we set $x$ equ $-1$, 3, or $-5$, then $x + 1$, $x - 3$, or $x + 5$, respectivel are factors of the polynomial. The following theorem can b proved.

> ***If a polynomial becomes 0 when $x = a$, then $x - a$ is a factor of the polynomial.***

*Example* 1. Test $x - 1$, $x + 1$, $x - 2$, $x + 2$, for factors $x^3 - 4x^2 + x + 6$.

If $x = 1$, the expression equals $1 - 4 + 1 + 6 = 4$. ence $x - 1$ is not a factor.

If $x = -1$, the expression equals $-1 - 4 - 1 + 6 = 0$. ence $x + 1$ is a factor.

If $x = 2$, the expression equals $8 - 16 + 2 + 6 = 0$. ence $x - 2$ is a factor.

If $x = -2$, the expression equals $-8 - 16 - 2 + 6 = 20$. ence $x + 2$ is not a factor.

*ample* 2. Factor $x^3 - 5x^2 - x + 5$.

First try to discover a factor by using the test. If we set = 1, the expression equals $1 - 5 - 1 + 5 = 0$. Hence - 1 is a factor.

$$
\begin{array}{r}
x^2-4x-5 \\
\hline
1)\,x^3-5x^2-x+5 \\
\quad x^3-\ x^2 \\
\hline
\quad -4x^2-x \\
\quad -4x^2+4x \\
\hline
\qquad -5x+5 \\
\qquad -5x+5 \\
\hline
\qquad\quad 0
\end{array}
$$

Long division shows that the other factor is $x^2-4x-5$. Therefore we have

$$
\begin{aligned}
x^3-5x^2-x+5 &= (x-1)\,(x^2-4x-5) \\
&= (x-1)(x+1)(x-5).
\end{aligned}
$$

## EXERCISES

Test to find if $x - 1$ and $x - 2$ are factors of the follow- polynomials:

1. $x^3-2x^2-x+2$. 2. $x^3+x^2-5x-2$. 3. $x^3-x^2-4x+4$.

Test to find if $a + 1$ and $a + 2$ are factors of the follow- polynomials:

4. $a^3+2a^2+a+2$. 5. $a^3+3a^2-a-3$. 6. $a^3+2a^2-a-2$.

7–12. Use long division to complete the factoring in Exs. 1-6.

Factor the following polynomials given that each has for a or one of $x - 1$, $x + 1$, $x - 2$, or $x + 2$.

13. $x^3 - 4x^2 + 4x - 1$.
14. $x^3 - 4x^2 + x + 6$.
15. $x^3 + 2x^2 + 5x + 4$.
16. $x^3 + 5x^2 - 2x - 16$.

## *Summary of Factoring*

The real test of your ability to factor comes in dealing with mixed sets of exercises involving all the types of factoring that you have studied. Before attacking such a set, it is advisable to recall the various types and the methods of procedure. The following table may be helpful for this purpose.

| *Type* | *Suggested By* | *Method* |
|---|---|---|
| *Common factor* | Factor common to every term | (C.F.) (second factor |
| *Trinomial* | Three terms | Intelligent guessing |
| *Difference of squares* | $(\ )^2 - (\ )^2$ | Use the formula: $x^2 - y^2 = (x-y)\ (x+y$ |
| *Incomplete squares* | At least fourth power; two terms are perfect squares; product term defective | Complete the square to get: $(\ )^2 - (\ )^2$ |
| *Sum and difference of cubes* | $(\ )^3 \pm (\ )^3$ | Use formulas: $x^3+y^3$ $=(x+y)\ (x^2-xy+y^2$ $x^3-y^3$ $=(x-y)\ (x^2+xy+y^2)$ |
| *Factor theorem* | Any polynomial, usually with at least four terms | If polynomial equals when $x=a$, then $x-$ is a factor. |

**EXERCISES**

ıctor:

1. $x^2 - 12x + 35$.
2. $x^2 - 81$.
3. $m^3 + 1$.
4. $a^3b - 4ab^3$.
5. $2x^2 + x - 6$.
6. $8a^3 - 27$.
7. $6ab + 3ac - 9ad$.
8. $25a^2 - 49$.
9. $125a^3 - 8$.
10. $6c^2 + 7c - 5$.
11. $r^4 - 1$.
12. $4x^3y^2 - 2x^2y^3$.
13. $75k^2 - 27$.
14. $8n^3 + 64$.
15. $2a^2 - 13a + 20$.
16. $24x^4 + 81x$.
17. $s^4 + s^2 + 1$.
18. $6a^4b^3 - 9a^3b^2 + 3a^2b$.
19. $40x^3 - 5$.
20. $p^4 - 5p^2 + 4$.
21. $x^4 - 5x^2y^2 + 4y^4$.
22. $6n^3 - 162$.
23. $kr^3 + k$.
24. $a^2b^2 - \frac{1}{4}$.
25. $3x^2 + 16x - 35$.
26. $7ax^3 + 28ax^2 + 28ax$.
27. $t^4 - 8t^2 + 16$.
28. $y^3 - 6y^2 + 9y$.
29. $1 - x^2y^4$.
30. $21a^2 - 23a + 6$.
31. $m^4n - 5m^3n^2 + 4m^2n^3$.
32. $r^4 - 4r^2$.
33. $1 - a^6$.
34. $k^4 - 3k^2 + 1$.
35. $5x^4y^5 + 40xy^2$.
36. $12x^2 - 8x - 7$.

## *Some Uses of Factoring*

In both arithmetic and algebra, frequent use is made of the thod of reducing fractions to their lowest terms. The re- :tion enables us to avoid working with fractions with iecessarily large terms; thus, we replace 17/51 by 1/3. The ic principle which brings about this reduction is the follow- .

***The terms of a fraction may be multiplied or divided by any number other than 0 without changing the value of the fraction.***

Study the following examples from arithmetic and algebra

| *Arithmetic* | *Algebra* |
|---|---|
| $\frac{24}{36} = \frac{2}{3}$ | $\frac{a^2x^3y}{axy^2} = \frac{ax^2}{y}$ |
| (Divide terms by 12) | (Divide terms by $axy$) |
| $\frac{\cancel{27}^{9}(18+73)}{\cancel{15}^{5}(18+73)} = \frac{9}{5}$ | $\frac{\cancel{km^2}^{m}(n+4)}{\cancel{k^2m}^{k}(n+4)} = \frac{m}{k}$ |
| (Divide terms by 3 and $18 + 73$) | (Divide terms by $km$ and $n + 4$) |

This process of dividing common factors from numerato and denominator can be used to simplify some problems o multiplication and division. Recall that

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} .$$

Usually the common factors, if any, are divided out ***befor*** the multiplication takes place.

*Examples.* The following examples illustrate the manner i which the work is carried out.

| *Arithmetic* | *Algebra* |
|---|---|
| $\overset{1}{\cancel{1}}\ \overset{1}{\cancel{3}}$<br>$\cancel{13}\ \cancel{18}$<br>$\cancel{24}\ \cancel{39}$<br>$4\ \ \cancel{3}$<br>$1$ | $\overset{1}{\cancel{a^2b^2}}\ \overset{a+b}{\cancel{(a+b)^2}}$<br>$\cancel{(a+b)}\ \cancel{a^2b^2}$<br>$1\ \ a$ |
| $\frac{4}{7} \cdot \frac{11}{12} \div \frac{22}{49}$<br>$\overset{1}{\cancel{4}}\ \overset{1}{\cancel{11}}\ \overset{7}{\cancel{49}}$<br>$=\cancel{7} \cdot \cancel{12} \cdot \cancel{22} = \frac{7}{6}$<br>$1\ \ 3\ \ 2$ | $\frac{x+3}{x^2-4} \cdot \frac{x^2+4x+4}{x+7} \div \frac{x^2+5x+6}{x^2+5x-14}$<br>$= \frac{\overset{1}{\cancel{x+3}}}{\cancel{(x+2)}(x-2)} \cdot \frac{\overset{1}{\cancel{(x+2)^2}}}{\cancel{x+7}} \cdot \frac{\overset{1}{\cancel{(x+7)}}\ \overset{1}{\cancel{(x-2)}}}{\cancel{(x+3)}\ \cancel{(x+2)}} = 1$<br>$1\ \ 1\ \ 1\ \ 1\ \ 1$ |

It appears that in algebra the expressions should be written factored form where possible. The common factors can en be divided out and the multiplication completed.

We have used the idea that division by 22/49, for example, the same as multiplication by its reciprocal 49/22. Why is is true?

* * *

ıe answer is that we are just using another form of the basic inciple on page 382. If we have any number $n$ divided by /49, then the basic principle gives

$$\frac{n}{\frac{22}{49}} = \frac{49n}{22}.$$

e simply multiply the terms of the complex fraction by 49. ou should note that *the basic principle provides a very conıient method of simplifying complex fractions both in arithmetic d algebra.* How would you simplify these fractions?

$$\frac{\frac{3}{4}}{\frac{5}{8}}, \quad \frac{\frac{2}{7}}{\frac{3}{2}}, \quad \frac{\frac{a}{b}}{\frac{1}{b}}, \quad \frac{\frac{c}{d}}{\frac{a}{d^2}}.$$

* * *

If you multiply the terms of the complex fractions by 8, 14, $b$ and $d^2$, respectively, you obtain $6/5$, $4/21$, $a$, $cd/a$.

### ORAL EXERCISES

State a common factor of each of the following:

1. 75, 125.
2. $ab$, $bc$.
3. $x^2y$, $xy^2$.
4. $kmn$, $mnr$.
5. $x-y$, $x^2-y^2$.
6. $m+n$, $m^2-n^2$.
7. $a^2-b^2$, $a^3-b^3$.
8. $x^2+2xy+y^2$, $x^3+y^3$.
9. $x^2-4$, $x^2+5x+6$.

### EXERCISES

*For Written Answers*

Reduce the following fractions to their lowest terms:

1. $\frac{45}{60}$.
2. $\frac{ax^2}{a^2x}$.
3. $\frac{120}{360}$.
4. $\frac{xyz}{ay}$.
5. $\frac{5280}{3600}$.
6. $\frac{12ab^2c}{4ab^3}$.
7. $\frac{m^2n(2m+n)}{n^2m(2m+n)}$.
8. $\frac{72(369+746)}{48(369+746)}$.
9. $\frac{(x-3)\ (x+2}{(x+3)\ (x+2}$
10. $\frac{a^2-b^2}{a+b}$.
11. $\frac{2x-4}{x^2-4}$.
12. $\frac{m^2-7m+1}{m^2-9}$
13. $\frac{18xy(x+y)}{9x+9y}$.
14. $\frac{10(x^2-2x-15)}{5(x^2+5x+6)}$.
15. $\frac{26(a^2-25)}{13(a+5)}$.

. $\dfrac{m-2}{m^3-8}$. 17. $\dfrac{4x^2-9}{8x^3-27}$. 18. $\dfrac{c^3+d^3}{c^4+c^2d^2+d^4}$.

e the basic principle on page 382 to simplify the following:

. $\dfrac{\frac{3}{5}}{\frac{7}{10}}$. 20. $\dfrac{\frac{1}{2}}{\frac{1}{3}}$. 21. $\dfrac{\frac{2}{3}}{\frac{5}{9}}$.

$\dfrac{\frac{x}{y}}{\frac{1}{y}}$. 23. $\dfrac{\frac{m}{n}}{\frac{3}{n^2}}$. 24. $\dfrac{\frac{a}{2}}{\frac{b}{3}}$.

$\frac{7}{8}\cdot\frac{12}{21}+\frac{5}{6}$. 26. $\dfrac{x^3y^2}{x+y}\cdot\dfrac{x^2-y^2}{x^2y^2}$.

$\frac{30}{75}\cdot\frac{25}{14}\div\frac{1}{7}$. 28. $\dfrac{a^2-16}{a+5}\cdot\dfrac{a^2+9a+20}{(a+4)^2}$.

$\dfrac{4x+8}{2x+6}\cdot\dfrac{5x+15}{5x+10}$. 30. $\dfrac{3k+2n}{7k-5n}\cdot\dfrac{7k-5n}{3k+2n}$.

$\dfrac{3a}{x-7}\cdot\dfrac{x^2-49}{12a}$.

$\dfrac{m^2+n^2+2mn}{m-n}\cdot\dfrac{m^2+n^2-2mn}{m+n}\cdot\dfrac{mn}{m^2-n^2}$.

$\dfrac{x^3+y^3}{x^2-y^2}\cdot\dfrac{x^2-2xy+y^2}{x^3-y^3}$.

$\dfrac{2a^2+a-1}{8a^3-1}\div\dfrac{a^2+4a+3}{a^2-9}$.

$\dfrac{m^2+mn+n^2}{m^4+m^2n^2+n^4}\cdot\dfrac{m^3+n^3}{m^2+2mn+n^2}$.

$\dfrac{m^2+2m+4}{m^3-8}\cdot\dfrac{m^2-4}{m^2-2m+4}\div\dfrac{m^2+4mn+4}{m^3+8}$.

37. $\dfrac{a^2-5a+6}{a^2+2a-15} \cdot \dfrac{a^2+12a+35}{a^2+a-42}$.

38. $\dfrac{3x^2y^3}{7m^2n^2} \cdot \dfrac{21m^3n}{9x^3y^3}$.

39. $\dfrac{x^3+3x^2y+3xy^2+y^3}{x^2-y^2} \cdot \dfrac{x-y}{(x+y)^2}$.

40. $\dfrac{2m^2+4m}{m^2+2m+4} \cdot \dfrac{m^3-8}{2m} \cdot \dfrac{1}{m^2-4}$.

41. $\dfrac{a^2-3a+2}{a^2-5a+4} \cdot \dfrac{a-4}{a-2}$.

42. $\left(\dfrac{x^2}{y^2} \div \dfrac{x^3}{y^3}\right) \cdot \dfrac{x^4-x}{y^3-y}$.

43. $\dfrac{25(k+m)^3}{3(k^2-m^2)} \div \dfrac{9(k+m)}{5(k-m)}$.

44. $\dfrac{x-3}{x+2} \cdot \dfrac{x^2-9x-22}{x^2-10x+21} \div \dfrac{x-11}{x^2-49}$.

45. $\dfrac{8x}{3x+1} \cdot \dfrac{9x^2-1}{x+5} \div \dfrac{24x^2}{3x+15}$.

46. $\dfrac{a^2-ab}{a+b} \cdot \dfrac{a^2+2ab+b^2}{a^2+ab} \div \dfrac{a^2-b^2}{a^2+b^2}$.

47. $\dfrac{4m^4+1}{2m^2+2m+1} \cdot \dfrac{2m-1}{8m^3-1}$.

48. $\dfrac{c+2}{c^2-4} \cdot \dfrac{c^4-10c^2+9}{c^2-9} \div \dfrac{c^2-1}{c^3-8}$.

## *Adding and Subtracting Fractions*

Let us consider two simple examples of the addition fractions.

| *Arithmetic* | *Algebra* |
|---|---|
| $\frac{2}{3}+\frac{1}{2}$ | $\frac{a}{b}+\frac{c}{d}$ |
| $=\frac{4}{6}+\frac{3}{6}=\frac{4+3}{6}=\frac{7}{6}$ | $=\frac{ad}{bd}+\frac{bc}{bd}=\frac{ad+bc}{bd}$ |

ow is the addition of the two fractions carried out?

* * *

The method involves bringing the fractions to a common enominator. This can be done by multiplying the terms of le fractions by suitable numbers; for example, we changed /3 and 1/2 into 4/6 and 3/6 by multiplying their terms by 2 id 3, respectively. The question of what number to use as a mmon denominator arises. In order to answer it, we need remember something about common multiples.

Recall that, since 7 and 3 are factors of 21, 21 is a ***multiple*** 7 and also of 3. In the same way, $x^2y$ is a multiple of $x^2$ d of $y$, $x^2-y^2$ is a multiple of $x+y$ and of $x-y$, and $^2+m$ is a multiple of $m$ and of $m+1$.

*If one number is a* ***factor*** *of another, then the second number is a* ***multiple*** *of the first.*

**EXERCISE**

State the converse of the foregoing statement. Is the nverse true?

* * *

Thus, since $x+y$ is a factor of $x^2-y^2$, it follows that

$x^2 - y^2$ is a multiple of $x + y$. Is $x^2 + y^2$ a multiple of $x + y$? Is $x^2 + y^2 + 2xy$ a multiple of $x + y$?

We are now ready to look into the question of common denominators. In the arithmetic example worked above, we chose 6 for our common denominator; 12, 18, 24, etc. would serve equally well. In the algebraic example above, $bd$ was chosen for a common denominator; $b^2d$ and many others would be acceptable. What is essential in the choice?

* * *

You see that ***the common denominator must be a multiple of the denominator of each fraction to be added or subtracted.*** Thus there are for a given problem many possible common denominators. A little thought will convince you that there is always one that is simpler than the others. It is sometimes called the *lowest common denominator* being the *lowest common multiple* of the denominators of the given fractions.

The first step shown in the two examples worked at the beginning of this section is usually omitted. You should always have it in mind, however, so as to understand the nature of the operation. Study the following examples carefully.

*Example* 1. Find the lowest common multiple of

$$x^2 - y^2, \text{ and } x^2 - 2xy + y^2.$$

Factoring the expressions, we obtain

$$x^2 - y^2 = (x - y)(x + y),$$
$$x^2 - 2xy + y^2 = (x - y)^2.$$

The lowest common multiple must include the factors $x - y$, $x + y$, and $(x - y)^2$. Therefore,

$$\text{L. C. M.} = (x + y)(x - y)^2.$$

*Example* 2. Simplify: $\dfrac{1}{x - 5} + \dfrac{2}{x + 5}$.

The simplest common multiple of $x - 5$ and $x + 5$ is $(x - 5)(x + 5)$ or $x^2 - 25$. Adopting this for our common denominator, we have

$$\frac{1(x + 5) + 2(x - 5)}{(x - 5)(x + 5)} = \frac{x + 5 + 2x - 10}{x^2 - 25} = \frac{3x - 5}{x^2 - 25}.$$

Now try the next example without further assistance. If you fail, get as much help as you need from the solution given below.

*Example* 3. Simplify: $\dfrac{3a}{a^2 + 3a - 10} - \dfrac{a}{a^2 + a - 6}$.

* * *

The first step is to factor the denominators. This gives

$$\frac{3a}{(a + 5)(a - 2)} - \frac{a}{(a + 3)(a - 2)}.$$

We could use $(a + 5)(a - 2)(a + 3)(a - 2)$ as our common denominator since it is a multiple of both given denominators. However, you see that it is needlessly large; $(a + 5)(a - 2)(a + 3)$ is a simpler common multiple. Proceeding as above, we get

$$\frac{a(a+3) - a(a+5)}{a+5)(a-2)(a+3)} = \frac{3a^2+9a-a^2-5a}{(a+5)(a-2)(a+3)} = \frac{2a(a+2)}{(a+5)(a-2)(a+3)}.$$

Try the next example by yourself. Get only as much help as you really need from the solution given below.

*Example* 4. Simplify: $\dfrac{m}{m^3 - m} + \dfrac{m^2 + 5m + 4}{m^2 + 3m - 4} - \dfrac{4}{2m + 2}$.

* * *

This example shows *the importance of making sure that each fraction is in its lowest terms before trying to add or subtract.* As in arithmetic, failure to bring to lowest terms almost always

results in needlessly long common denominators. Factoring shows us what to divide out. We have

$$\frac{\overset{1}{\cancel{m}}}{\underset{1}{\cancel{m}(m^2-1)}} + \frac{1}{\underset{1}{\cancel{(m-1)}}\,\underset{1}{\cancel{(m+1)}}}\,\cancel{(m+1)} - \frac{\overset{2}{\cancel{4}}}{\underset{1}{\cancel{2}}(m+1)} = \frac{1}{m^2-1} + \frac{m+1}{m-1} - \frac{2}{m+1},$$

$$= \frac{1+(m+1)\,(m+1)-2(m-1)}{m^2-1},$$

$$= \frac{1+m^2+2m+1-2m+2}{m^2-1},$$

$$= \frac{m^2+4}{m^2-1}.$$

**Warning: A Popular Pitfall.** Be careful to distinguish between the following two operations:

(1) . . . $\frac{a}{x} \cdot \frac{x}{b}$ and (2) . . . $\frac{a}{x} + \frac{x}{b}$.

Can the $x$ be divided out in (1)? In (2)? What is the difference?

* * *

(1) involves multiplication so that we have

$$\frac{a}{x} \cdot \frac{x}{b} = \frac{ax}{bx}.$$

Therefore, $x$ may be divided from the terms by virtue of our basic principle. The situation is entirely different in (2). The operation is addition which gives us that

$$\frac{a}{x} + \frac{x}{b} = \frac{ab + x^2}{bx}.$$

Thus, $x$ is not a factor of both terms and cannot be divided out. Failure to tell these operations apart may cause you to make many blunders.

## EXERCISES

Find the lowest common multiple for each of the following sets of numbers.

1. $4, 6, 12$.
2. $18, 24, 36$.
3. $ax, bx, cx$.
4. $3y, 6y^2, 9y^3$.
5. $km^2, kn^2$.
6. $bx^2, b^2x^3$.
7. $x + y, x^2 - {}^2y$.
8. $x - y, x^2 + y^2 - 2xy$.
9. $a + b, c + d$.
10. $(a+5)\,(a-2),\ (a+3)\,(a-2)$.
11. $(x + y)^2, x^2 - y^2$.
12. $bx^2, x(x - y), b(x - y)$.
13. $a^2 - b^2, a^3 - b^3$.
14. $m^2 + 4m + 4, m^3 + 8$.

Simplify the following by performing the indicated additions and subtractions.

15. $\frac{1}{2} + \frac{1}{3}$.
16. $\frac{1}{x} + \frac{1}{y}$.
17. $\frac{3}{4} - \frac{5}{12}$.
18. $\frac{3}{b} - \frac{5}{bc}$.
19. $\frac{5}{6} + \frac{7}{15}$.
20. $\frac{5}{xy} + \frac{7}{yz}$.
21. $\frac{2}{3} + \frac{7}{14} - \frac{5}{21}$.
22. $\frac{2}{m} + \frac{7}{kn} - \frac{5}{mn}$.
23. $\frac{13}{18} - \frac{7}{12} + \frac{11}{30}$.
24. $\frac{13}{xy} - \frac{7}{xz} + \frac{11}{xa}$.
25. $\frac{a + b}{4} + \frac{a - b}{5}$.
26. $\frac{x - y}{2} + \frac{x + y}{6} - \frac{2x - y}{8}$.
27. $\frac{3}{x + y} - \frac{2}{x - y}$.
28. $\frac{y^2}{x^2 - y^2} + \frac{y}{x + y}$.

29. $\dfrac{a}{a+b} - \dfrac{a^2}{(a+b)^2}$.

30. $\dfrac{1}{(x+y)^2} + \dfrac{1}{x^2-y^2}$.

31. $\dfrac{x-y}{x+y} - \dfrac{x+y}{x-y}$.

32. $\dfrac{a-b}{a} + \dfrac{2b}{a-b}$.

33. $\dfrac{1}{x-y} + \dfrac{x}{x^3-y^3}$.

34. $\dfrac{1}{a+b} - \dfrac{b}{a^3+b^3}$.

35. $\dfrac{1}{a-b} + \dfrac{1}{a+b} - \dfrac{2a}{a^2-b^2}$.

36. $\dfrac{x-1}{x-2} + \dfrac{x-3}{x-4}$.

37. $\dfrac{x}{x^2-x} + \dfrac{1}{x+1}$.

38. $\dfrac{m^2}{m^2-n^2} - \dfrac{m^2}{m^2+mn}$.

39. $\dfrac{7}{x^2+xy} - \dfrac{5}{x^2-xy}$.

40. $\dfrac{1}{a^2+3a+2} + \dfrac{1}{a^2+4a+3}$

41. $\dfrac{x+y}{x^2-y^2} - \dfrac{1}{x-y}$.

42. $\dfrac{a+b}{a^2+2ab+b^2} + \dfrac{a-b}{a^2-2ab+b^2} - \dfrac{2a^2}{a^3-ab^2}$

43. $\dfrac{4a^2-b^2}{2a^2+ab} - \dfrac{2a-b}{b}$.

44. $\dfrac{5}{x-3} - \dfrac{5}{x+3} - \dfrac{30}{x^2-9}$

45. $\dfrac{a^2-a-2}{a^2-2a-3} - \dfrac{a^2-5a+6}{a^2-7a+10}$.

46. $\dfrac{(2x+3)^2}{2x^2+13x+15} + \dfrac{x^2+x-30}{x^2-10x+25} - \dfrac{3x^2+4x}{x^2-25}$.

47. $\dfrac{1}{a^2+ab+b^2} - \dfrac{1}{a^4+a^2b^2+b^4}$.

48. $\dfrac{1}{2x-3} - \dfrac{2x+3}{4x^2+6x+9} + \dfrac{5}{8x^3-27}$.

## *Further Practice with Fractions*

In transforming expressions involving fractions, the following device is sometimes helpful.

**Use of the multiplier $-1$.** The basic principle of fractions permits us to multiply the terms of a fraction by $-1$. Thus

$$\frac{3}{4} = \frac{-3}{-4}, \quad \text{and} \quad \frac{-3}{4} = \frac{3}{-4}.$$

From the rule of signs for division (page 7), we have

$$\frac{-3}{4} = -\frac{3}{4} = \frac{3}{-4}.$$

It follows that *the value of a fraction remains unchanged*

(1) *if both of its terms are multiplied by* $-1$, *or*

(2) *if one term is multiplied by* $-1$, *and the sign before the fraction is changed.*

These uses of $-1$ as a multiplier sometimes enable us to change a fraction into a more convenient form.

*Example* 1. (*a*) $$-\frac{m}{-2} = +\frac{m}{2} = -\frac{-m}{2}.$$

(*b*) $$-\frac{a-b}{y-x} = +\frac{a-b}{x-y} = -\frac{b-a}{x-y} = +\frac{b-a}{y-x}.$$

*Example* 2. Simplify: $\dfrac{1}{m-2n} - \dfrac{2m}{4n^2 - m^2}$.

For the purpose of finding the simplest common denominator, the denominator of the second fraction would be more conveniently written as $m^2 - 4n^2$, which is $-(4n^2 - m^2)$. Therefore we multiply the denominator of the second fraction by $-1$ and change the sign before the fraction. This gives

$$\frac{1}{m-2n} + \frac{2m}{m^2 - 4n^2} = \frac{(m+2n) + 2m}{(m+2n)(m-2n)},$$

$$= \frac{3m + 2n}{m^2 - 4n^2}.$$

## EXERCISES

### *For Oral Answers*

**1.** Express each of the following as a fraction with no minus signs in its terms:

$(a)\ \frac{-4}{5}$, $(b)\ \frac{4}{-5}$, $(c)\ \frac{-4}{-5}$, $(d)\ \frac{-a}{n}$, $(e)\ \frac{-c}{-d}$, $(f)\ \frac{c}{-d}$.

**2.** Express each of the following as a fraction with a plus sign in front:

$(a)\ -\frac{-a}{b}$, $(b)\ -\frac{-m}{-n}$, $(c)\ -\frac{x}{-y}$, $(d)\ -\frac{b-a}{c}$,

$(e)\ -\frac{x}{y-z}$, $(f)\ -\frac{m-n}{-k}$, $(g)\ -\frac{-c}{a-b}$, $(h)\ -\frac{b-a}{y-x}$,

$(i)\ -\frac{2}{p-q}$, $(j)\ -\frac{2(a-b)}{x-y}$.

**3.** Express each of the following as a fraction with a minus sign in front:

$(a)\ \frac{-7}{6}$, $(b)\ \frac{d}{-c}$, $(c)\ \frac{x}{y}$, $(d)\ \frac{a-b}{c}$, $(e)\ \frac{m}{k-n}$,

$(f)\ \frac{-x}{y-z}$, $(g)\ \frac{x-y}{a-b}$, $(h)\ \frac{a(b-c)}{m-n}$, $(i)\ \frac{5(y-z)}{-3(a-b)}$.

### *For Written Answers*

Perform the indicated additions and subtractions.

**4.** $\frac{1}{a-b}+\frac{1}{b-a}$.

**5.** $\frac{6}{3x-y}-\frac{4}{y-3x}$.

**6.** $\frac{1}{c+2d}-\frac{2d}{4d^2-c^2}$.

**7.** $\frac{m^2}{n^2-m^2}+\frac{m}{m-n}$.

8. $\dfrac{1}{x-y} + \dfrac{1}{x+y} + \dfrac{2x}{y^2-x^2}$.

9. $\dfrac{3}{x+2} + \dfrac{3}{2-x} + \dfrac{12}{x^2-4}$.

10. $\dfrac{a+b}{3a-2b} - \dfrac{a-b}{2b-3a}$.

11. $\dfrac{4}{k-1} - \dfrac{3}{k+1} + \dfrac{k-3}{1-k^2}$.

**Equations involving fractions.** In Chapter I, we met some equations involving numerical fractions. The basic principles listed on page 20 enable us to solve such equations quite readily. Moreover, these principles are all that we need for the solution of equations with literal fractions like those in the examples below. Study these examples carefully. Note that the simplest solution requires you to find the L.C.M. of the denominators of the fractions which appear.

*Example* 1. Solve: $\dfrac{4}{x} - \dfrac{9}{2x} = 1 - \dfrac{2}{3x}$.

The fractions can be easily eliminated by multiplying both sides of the equation by $6x$ which is the L.C.M. of $x$, $2x$, and $3x$. This gives

$$24 - 27 = 6x - 4$$

whence
$$-3 = 6x - 4,$$
$$1 = 6x,$$

and
$$x = 1/6.$$

Verify this answer.

*Example* 2. Solve: $\dfrac{2a}{a-1} - \dfrac{3}{a+1} = 3.$

The L.C.M. of $a - 1$ and $a + 1$ is $a^2 - 1$. Hence, we multiply both sides of the equation by $a^2 - 1$, which gives

$$2a(a + 1) - 3(a - 1) = 3(a^2 - 1).$$

Therefore we have

$$2a^2 + 2a - 3a + 3 = 3a^2 - 3,$$
$$2a^2 - a = 3a^2 - 6,$$

and
$$a^2 + a - 6 = 0.$$

Hence $(a + 3)(a - 2) = 0$, and $a = -3$ or $2$.

Show that both values of $a$ satisfy the given equation.

## EXERCISES

In each of the following equations, solve for the unknown and verify your answer by substitution.

1. $\frac{x}{2} - 3 = \frac{x}{3} + 1.$

2. $\frac{2x + 1}{3} = \frac{2x + 4}{4}.$

3. $\frac{1.2 - 0.3x}{0.2} = \frac{3.2 - 0.6x}{1.2}.$

4. $\frac{a - 2}{2} - \frac{a + 2}{4} = \frac{a - 3}{3}.$

5. $\frac{2(x - 7)}{3} = \frac{3x}{5} - 1.$

6. $\frac{1}{5}(x + 1) - \frac{2}{3}(x - 2) = 2.$

7. $\frac{1}{y} + \frac{2}{y} = 3 - \frac{3}{y}.$

8. $\frac{3}{x} - \frac{1}{4x} = \frac{11}{16}.$

9. $\frac{x + 5}{2x} - \frac{3x - 1}{x} = 1.$

10. $\frac{7}{2x} - \frac{3}{4x} + \frac{1}{x} = \frac{1}{4}.$

11. $\frac{4}{x} - \frac{x - 5}{3x} = \frac{16}{3}.$

12. $\frac{5x}{2} - (x - 3) = \frac{3}{2}(4 - x) - \frac{9}{2}.$

13. $\frac{4}{x - 2} = 1.$

14. $\frac{3(a - 1)}{a + 7} = -\frac{3}{5}.$

15. $\frac{5}{x} - 3 = \frac{9}{4x} - \frac{1}{4}.$

16. $\frac{2}{x - 1} = \frac{3}{x + 2}.$

17. $\frac{4z}{z-1} - \frac{3}{z+1} = 4.$

18. $\frac{2}{x-1} + \frac{3}{x} = \frac{5}{x+1}.$

19. $\frac{x}{x + 1} + 2 = \frac{3x}{x + 2}.$

20. $\frac{5}{x - 4} + \frac{2}{x + 4} = \frac{-2}{x^2 - 16}.$

21. $\frac{a}{a - 2} - 1 = \frac{4a}{a + 3}.$

22. $\frac{3}{a - 1} + 2 = \frac{3}{a + 1} + \frac{12}{a^2-1}.$

Solve the following problems by introducing only one unknown number.

**23.** The sum of one-half of a number, one-third of the number and one-quarter of the number equals 65. Find the number.

**24.** Find two consecutive numbers such that one-half of the first increased by one-third of the second equals 17.

**25.** When 8 is subtracted from two-thirds of a certain number the result is the same as when 4 is added to one-sixth of the number. Find the number.

**26.** The denominator of a fraction is 3 more than the numerator. If both terms of the fraction are increased by 7, its value becomes $\frac{3}{4}$. Find the fraction.

**27.** The denominator of a fraction is 2 less than the numerator. When 1/3 is added to the fraction the result is the same as when 4/3 is subtracted from twice the fraction. Find the fraction.

**28.** The sum of a certain number and its reciprocal is 2. Find the number. (NOTE: the *reciprocal* of $x$ is $1/x$; thus 5 and 1/5 are reciprocals.)

**29.** Three times a certain number is added to twice the reciprocal of the number, giving the sum 5.5. Find the number.

**30.** The sum of the reciprocal, one-half the reciprocal, and one-third of the reciprocal of a certain number is 1. Find the number.

**31.** When the reciprocal of a certain number is decreased by one-quarter of the reciprocal, the result equals 1 increased by twice the reciprocal. Find the number.

**32.** When the quotient of 2 divided by a certain number is increased by the quotient of 3 divided by 2 more than the number, the result is 1. Find the number.

**33.** Divide 100 into two parts so that the quotient of the first part divided by 5 equals 18 increased by the quotient of the second part divided by 10.

**34.** A pupil was told to add 6 to a certain number and divide the result by 7. In error, he added 7 to the number and divided by 6, getting the correct answer. What was the number?

**35.** When 2 is added to the quotient of 3 divided by a certain number, the result equals the quotient obtained by dividing 10 by 2 less than the given number. What is the number?

**36.** An automobile travelling at an average speed of $r$ miles per hour for $t$ hours covers $d$ miles, where $d = rt$. Use this formula to answer the following:

A car travels 100 miles in $x$ hours. What is the average speed? A car travels 100 miles at an average speed of $x$ m.p.h. What was the time taken? What distance is covered in travelling 5 hours at an average speed of $x$ m.p.h.? What distance is covered in travelling $x$ hours at an average speed of 40 m.p.h.? What time is required to travel $x$ miles at an average speed of 30 m.p.h.? What is the average speed for a journey of $x$ miles in 10 hours?

**37.** A car travels from $A$ to $B$, a distance of 240 miles, and returns. The average speed on the return journey is double that on the out-going journey, and the total time required for travelling is 9 hours. Find the average speed on the out-going trip.

**38.** A man walks from Smithburg to Jonesville and back, without stopping, in 6 hours. If the two towns are 8 miles apart and if his average speed returning is 2 miles per hour more than his average speed going, find his average speed going.

## REVIEW TESTS ON CHAPTER XVIII

### *Test No. 1*

Factor the following:

1. $2x^3-4x^2y$.
2. $m^2+7m+6$.
3. $x^2-y^2$.
4. $2x^2+x-6$.
5. $2n^2-8$.
6. $9a^2b-6ab^2+3ab$.
7. $x^3-64$.
8. $27m^3+1$.
9. $x^4-10x^2+9$.

Expand the following:

10. $(a+2b)^2$.
11. $(2x-y)^2$.
12. $(a+b+2c)^2$.
13. $(m+n)^3$.
14. $(r-3)^3$.
15. $(x+3y)^3$.

Simplify the following by performing the indicated multiplications and collecting like terms.

16. $(m+n)^2-3(m+n)(m-n)+2(m-n)^2$.
17. $(x+y-z)^2+(x+y)^2-(y-z)^2-(x-z)^2$.
18. $(x-2y)^3-(2y-x)^3$.
19. $3(a-2b)^2-2(2a-b)^2-10(2a+b)^2$.
20. Multiply $2m^2-3mn+4n^2$ by $m^2+mn-2n^2$ and check your work when $m=1, n=2$.
21. Divide $27a^3+9a^2-3a-10$ by $3a-2$.
22. Divide $x^3-6x^2+12x-8$ by $x^2-4x+4$.

Use the basic principle to simplify the following fractions:

23. $\dfrac{72x^3y^2}{90xy^3}$.
24. $\dfrac{3x^2-3y^2}{6(x^2+2xy+y^2)}$.
25. $\dfrac{\frac{4}{7}}{3}$.
26. $\dfrac{5}{\frac{x-y}{x+y}}$.
27. Simplify: $\dfrac{x^2-x-20}{x^2-25}\cdot\dfrac{x^2+5x}{x+1}\div\dfrac{x^2+2x-8}{x^2-x-2}$.

**28.** What is the lowest common multiple of $2(x^2 - y^2)$, $4(x - y)^2$, and $6(x + y)$?

**29.** Simplify: $\dfrac{6}{x - y} - \dfrac{5x}{x^2 - xy}$.

**30.** Simplify: $\dfrac{x - y}{xy} + \dfrac{y - z}{yz} + \dfrac{z - x}{zx}$.

### *Test No. 2*

Factor the following:

**1.** $x^4 - 9x^2$. **2.** $9x^2 - 30xy + 25y^2$. **3.** $6x^3 - 3x^2y$.

**4.** $3a^3 + 81$. **5.** $m^4 - 8m$. **6.** $t^4 + t^2 + 1$.

**7.** If $x = a + b$ and $y = a - b$, find, in terms of $a$ and $b$, the value of (i) $x^2 + y^2$, (ii) $x^2 - y^2$.

Simplify the following by performing the indicated multiplications and collecting like terms.

**8.** $3(2x - 3y)^2 - 5(2x + 3y)(2x - 3y) + 2x(4x - y)$.

**9.** $(3a - 1)(a^2 - 2a + 3) - (a + 2)(3a^2 + 7a - 5)$.

**10.** $(x - 2y - 3z)^2 + (x + y - z)^2 - (x + 2y + 3z)^2$.

**11.** Divide $4x^3 + 7x^2 - 3x - 15$ by $4x - 5$.

**12.** Divide $6x^3 - x^2 - 14x + 3$ by $3x^2 + 4x - 1$.

**13.** Simplify: $\dfrac{9x^2 - 16}{x^2 - 4} \cdot \dfrac{x + 2}{3x - 4}$.

**14.** Simplify: $\dfrac{m^2 - 8m - 9}{m^2 - 17m + 72} \cdot \dfrac{m^2 - 9m + 8}{m^2 + 4m - 5} \div \dfrac{m^2 - 1}{m^2 - 25}$.

**15.** Simplify: $\dfrac{1}{x + y} - \dfrac{1}{y - x} - \dfrac{2x}{x^2 - y^2}$

16. Simplify: $\dfrac{1}{a^2 + 3a - 10} - \dfrac{1}{a^2 + a - 6}$.

17. Solve: $\dfrac{x + 1}{3} + \dfrac{x - 1}{2} = 4$.

18. Find two consecutive numbers such that one-fifth of the first increased by one-third of the second equals 27.

19. Solve: $\dfrac{2x}{x - 3} = \dfrac{2x + 4}{x} + 5$.

20. What number must be added to the numerator and the denominator of the fraction 4/7 so that the resulting fraction equals 6/7?

### *Test No. 3*

Express the following in the simplest factored form:

1. $1 - 4x^2$. 2. $b - 9ba^2$. 3. $6x^2 + 3x - 3$.

4. $a^3b - 16ab^3$. 5. $10m^4n^2 + 15m^3n^2 - 10m^2n^2$. 6. $k^2 - \dfrac{1}{9}$.

7. $3x^4y + 24xy^4$. 8. $x^3 - 3x^2 + 3x - 1$. 9. $(x - 2y)^2 - 25z^2$.

10. Simplify: $(3k - 2n)^3 + (3k + 2n)^3$.

11. Simplify:
$(3x - 1)(2x + 3)(4x - 5) - (6x + 5)(4x - 1)(x - 3)$.

12. Divide $39x - 46x^2 - 9 + 16x^3$ by $8x - 3$. Check your work when $x = 1$.

13. Divide $8a^3 + 12a + 7$ by $2a + 1$.

14. Multiply $5x^2 + 3xy - 4y^2$ by $2x^2 - xy + y^2$. Check your work when $x = -2$, $y = 1$.

Use the basic principle to simplify the following fractions:

15. $\dfrac{\frac{3}{5}}{\frac{2}{3}}$.

16. $\dfrac{\frac{a}{c^2}}{\frac{b}{c}}$.

17. $\dfrac{15m}{m^2-36} \cdot \dfrac{m-6}{5}$.

18. $\dfrac{x^2-7x+12}{x^2+2x-15} \cdot \dfrac{x^2+8x+15}{x^2+x-20}$

19. Simplify: $\dfrac{2x+y}{5} + \dfrac{x-2y}{10} - \dfrac{3y-x}{6}$.

20. Simplify: $\dfrac{2n}{2m+2n} - \dfrac{m}{m+n} - \dfrac{m-n}{n^2-m^2}$.

21. Solve: $\dfrac{x+1}{3x} + \dfrac{4-2x}{6x} = \dfrac{x-1}{2x}$. Verify your answer by substitution.

22. Solve: $\dfrac{a+3}{a-1} = 2 - \dfrac{a+1}{a-3}$. Verify your answer.

23. Divide 360 into two parts such that when the quotient of the first part divided by 10 is diminished by the quotient of the second part divided by 12, the remainder is 14.

### *Test No. 4*

Factor the following and verify by multiplication:

1. $4z^2 - 15z - 4$.
2. $(x+y)^2 - z^2$.
3. $125a^3 - 8b^3$.
4. $4m^4 - 5m^2 + 1$.
5. $40c^3 - 135d^3$.
6. $x^3 + x^2 + x + 1$.

7. Solve and verify: $\dfrac{1}{z-1} + \dfrac{1}{z-2} = \dfrac{5}{z^2-3z+2}$.

8. What number increased by twice its reciprocal equals 4.5?

9. If $x = a + b + c$ and $y = a - b - c$, evaluate $x^2 + y^2 + 2xy$ in terms of $a$, $b$, and $c$.

10. Simplify: $(5m+4n)^2-(5m+4n)(5m-4n)+(5m-4n)^2$.

11. Simplify: $(3x - 2y - 4z)^2 - (-3x + 2y + 4z)^2$.

Use the basic principle to simplify the following fractions:

12. $\dfrac{\frac{11}{24}}{\frac{}{5}}$.

13. $\dfrac{\frac{ax}{b}}{\frac{}{y}}$.

14. $\left(\dfrac{x^2+y^2-2xy}{x+y} \div \dfrac{x-y}{x^2+y^2+2xy}\right) \dfrac{x}{x^2-y^2}$.

15. $\dfrac{2a^2+11a+12}{4a^2-9} \div \dfrac{a^2-16}{2a^2-11a+12}$.

16. $\dfrac{m^2n^2-5mn}{4mn+20} \cdot \dfrac{m^2n^2+10mn+25}{m^2n^2-25} \div \dfrac{m^3n^3}{16}$.

17. $\dfrac{(r+s)^2}{(c+d)^2} \div \dfrac{(r+s)^3}{(c+d)^3}$.

18. Simplify: $\dfrac{a-b}{a} + \dfrac{a+b}{b} - \dfrac{a^2-b^2}{ab}$.

19. Simplify: $\dfrac{x+3}{x-5} - \dfrac{x-4}{x+2}$.

20. Simplify: $\dfrac{k+5}{k-3} + \dfrac{k+3}{5-k} + \dfrac{18}{k^2-8k+15}$.

21. What is the remainder when $2x^3 - 4x + 7$ is divided by $x - 2$?

22. The denominator of a certain fraction exceeds the numerater by 3. If 15 is added to the numerator of the fraction, its value becomes equal to the reciprocal of the original fraction. What is the original fraction?

23. A cyclist travels a distance of 12 miles and back, making the return trip in 3 hours. If his average speed on the return was twice his average speed going, find his average rate going.

## *Test No. 5*

Factor:

1. $25x^2-4y^2$.
2. $a^2b^2-c^2$.
3. $x^3y^3-z^3$.
4. $12x^2+8x-15$.
5. $1+1000r^3s^3t^3$.
6. $a^4+2a^2b^2+9b^4$.
7. $a^3-4a^2+a+6$.
8. $24x^2+43x+18$.
9. $(x+2y)^2-9z^2$.
10. Simplify: $(3a-4b)(5a+3b)-(7a-b)(2a+5b)+(3b-a)(a+3b)$.
11. Simplify: $(2x+1)^3-(3x-2)^3+(x-1)(19x+9)$.
12. Find the product of $3x^2+xy-4y^2$, $2x-y$, and $-x-y$. Check your work when $x=-1$, $y=-2$.
13. Find the value of $a$ which makes $3x^3-5x+a$ exactly divisible by $x-1$.
14. Divide $8a^3-b^3$ by $2a-b$.

Simplify the following:

15. $\dfrac{6x^2-x-2}{x-1}\cdot\dfrac{5x-5}{9x^2-4}\div\dfrac{2x+1}{3x+2}$.
16. $\dfrac{4}{a+2}+\dfrac{3}{2a-4}+\dfrac{5a}{4-a^2}$.
17. $\dfrac{m^2+mn}{m-n}\cdot\dfrac{m^2-mn+n^2}{m^2+mn}\div\dfrac{m^3+n^3}{m^2-n^2}$.

18. $\dfrac{1}{3(x-y)} - \dfrac{1}{3(x+y)} + \dfrac{2y}{y^2-x^2}$.

19. $\dfrac{c}{c^3-c} + \dfrac{c^2+5c+4}{c^2+3c-4} - \dfrac{4}{2c+2}$.

20. $\dfrac{m^2+4mn-3n^2}{m^2-9n^2} - \dfrac{2mn}{2m^2+6mn}$.

21. Solve and verify: $\dfrac{x-1}{5} - \dfrac{3x-1}{8} = 7 + \dfrac{1-5x}{6}$.

22. Solve and verify: $\dfrac{4x}{x+1} - \dfrac{x}{x-2} = 3$.

23. Solve and verify: $\dfrac{1}{x+1} + \dfrac{2}{x-2} = \dfrac{3}{x+3}$.

24. The denominator of a fraction is 9 more than the numerator. When the fraction is reduced to its lowest terms it becomes 2/3. Find the original fraction.

25. A pupil was instructed to divide 48 by a certain number. Instead he divided 48 by a number which exceeded the given number by 2. If his answer was two less than the correct answer, find the given number.

## TABLE I

Amount of 1 at the end of $n$ periods

$(1 + i)^n$

| $n$ | $\frac{3}{4}\%$ | $1\%$ | $1\frac{1}{4}\%$ | $1\frac{1}{2}\%$ | $1\frac{3}{4}\%$ | $2\%$ | $2\frac{1}{2}\%$ | $n$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 1.007500 | 1.010000 | 1.012500 | 1.015000 | 1.017500 | 1.020000 | 1.025000 | 1 |
| 2 | 1.015056 | 1.020100 | 1.025156 | 1.030225 | 1.035306 | 1.040400 | 1.050625 | 2 |
| 3 | 1.022669 | 1.030301 | 1.037971 | 1.045678 | 1.053424 | 1.061208 | 1.076891 | 3 |
| 4 | 1.030339 | 1.040604 | 1.050945 | 1.061364 | 1.071859 | 1.082432 | 1.103813 | 4 |
| 5 | 1.038067 | 1.051010 | 1.064082 | 1.077284 | 1.090617 | 1.104081 | 1.131408 | 5 |
| 6 | 1.045852 | 1.061520 | 1.077383 | 1.093443 | 1.109702 | 1.126162 | 1.159693 | 6 |
| 7 | 1.053696 | 1.072135 | 1.090850 | 1.109845 | 1.129122 | 1.148686 | 1.188686 | 7 |
| 8 | 1.061599 | 1.082857 | 1.104486 | 1.126493 | 1.148882 | 1.171659 | 1.218403 | 8 |
| 9 | 1.069561 | 1.093685 | 1.118292 | 1.143390 | 1.168987 | 1.195093 | 1.248863 | 9 |
| 10 | 1.077583 | 1.104622 | 1.132271 | 1.160541 | 1.189444 | 1.218994 | 1.280085 | 10 |
| 11 | 1.085664 | 1.115668 | 1.146424 | 1.177949 | 1.210260 | 1.243374 | 1.312087 | 11 |
| 12 | 1.093807 | 1.126825 | 1.160755 | 1.195618 | 1.231439 | 1.268242 | 1.344889 | 12 |
| 13 | 1.102010 | 1.138093 | 1.175264 | 1.213552 | 1.252990 | 1.293607 | 1.378511 | 13 |
| 14 | 1.110276 | 1.149474 | 1.189955 | 1.231756 | 1.274917 | 1.319479 | 1.412974 | 14 |
| 15 | 1.118603 | 1.160969 | 1.204829 | 1.250232 | 1.297228 | 1.345868 | 1.448298 | 15 |
| 16 | 1.126992 | 1.172579 | 1.219890 | 1.268986 | 1.319929 | 1.372786 | 1.484506 | 16 |
| 17 | 1.135445 | 1.184304 | 1.235138 | 1.288020 | 1.343028 | 1.400241 | 1.521618 | 17 |
| 18 | 1.143960 | 1.196147 | 1.250577 | 1.307341 | 1.366531 | 1.428246 | 1.559659 | 18 |
| 19 | 1.152540 | 1.208109 | 1.266210 | 1.326951 | 1.390445 | 1.456811 | 1.598650 | 19 |
| 20 | 1.161184 | 1.220190 | 1.282037 | 1.346855 | 1.414778 | 1.485947 | 1.638616 | 20 |
| 21 | 1.169893 | 1.232392 | 1.298063 | 1.367058 | 1.439537 | 1.515666 | 1.679582 | 21 |
| 22 | 1.178667 | 1.244716 | 1.314288 | 1.387564 | 1.464729 | 1.545980 | 1.721571 | 22 |
| 23 | 1.187507 | 1.257163 | 1.330717 | 1.408377 | 1.490361 | 1.576899 | 1.764611 | 23 |
| 24 | 1.196414 | 1.269735 | 1.347351 | 1.429503 | 1.516443 | 1.608437 | 1.808726 | 24 |
| 25 | 1.205387 | 1.282432 | 1.364193 | 1.450945 | 1.542981 | 1.640606 | 1.853944 | 25 |
| 26 | 1.214427 | 1.295256 | 1.381245 | 1.472710 | 1.569983 | 1.673418 | 1.900293 | 26 |
| 27 | 1.223535 | 1.308209 | 1.398511 | 1.494800 | 1.597457 | 1.706886 | 1.947800 | 27 |
| 28 | 1.232712 | 1.321291 | 1.415992 | 1.517222 | 1.625413 | 1.741024 | 1.996495 | 28 |
| 29 | 1.241957 | 1.334504 | 1.433692 | 1.539981 | 1.653858 | 1.775845 | 2.046407 | 29 |
| 30 | 1.251272 | 1.347849 | 1.451613 | 1.563080 | 1.682800 | 1.811362 | 2.097568 | 30 |
| 31 | 1.260656 | 1.361327 | 1.469759 | 1.586526 | 1.712249 | 1.847589 | 2.150007 | 31 |
| 32 | 1.270111 | 1.374941 | 1.488131 | 1.610324 | 1.742213 | 1.884541 | 2.203757 | 32 |
| 33 | 1.279637 | 1.388690 | 1.506732 | 1.634479 | 1.772702 | 1.922231 | 2.258851 | 33 |
| 34 | 1.289234 | 1.402577 | 1.525566 | 1.658996 | 1.803725 | 1.960676 | 2.315322 | 34 |
| 35 | 1.298904 | 1.416603 | 1.544636 | 1.683881 | 1.835290 | 1.999890 | 2.373205 | 35 |
| 36 | 1.308645 | 1.430769 | 1.563944 | 1.709140 | 1.867407 | 2.039887 | 2.432535 | 36 |
| 37 | 1.318460 | 1.445076 | 1.583493 | 1.734777 | 1.900087 | 2.080685 | 2.493349 | 37 |
| 38 | 1.328349 | 1.459527 | 1.603287 | 1.760798 | 1.933338 | 2.122299 | 2.555682 | 38 |
| 39 | 1.338311 | 1.474123 | 1.623328 | 1.787210 | 1.967172 | 2.164745 | 2.619574 | 39 |
| 40 | 1.348349 | 1.488864 | 1.643619 | 1.814018 | 2.001597 | 2.208040 | 2.685064 | 40 |

## TABLE I

Amount of 1 at the end of $n$ periods

$(1 + i)^n$

| $n$ | 3% | 3½% | 4% | 4½% | 5% | 5½% | 6% | $n$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 1.030000 | 1.035000 | 1.040000 | 1.045000 | 1.050000 | 1.055000 | 1.060000 | 1 |
| 2 | 1.060900 | 1.071225 | 1.081600 | 1.092025 | 1.102500 | 1.113025 | 1.123600 | 2 |
| 3 | 1.092727 | 1.108718 | 1.124864 | 1.141166 | 1.157625 | 1.174241 | 1.191016 | 3 |
| 4 | 1.125509 | 1.147523 | 1.169859 | 1.192519 | 1.215506 | 1.238825 | 1.262477 | 4 |
| 5 | 1.159274 | 1.187686 | 1.216653 | 1.246182 | 1.276282 | 1.306960 | 1.338226 | 5 |
| 6 | 1.194052 | 1.229255 | 1.265319 | 1.302260 | 1.340096 | 1.378843 | 1.418519 | 6 |
| 7 | 1.229874 | 1.272279 | 1.315932 | 1.360862 | 1.407100 | 1.454679 | 1.503630 | 7 |
| 8 | 1.266770 | 1.316809 | 1.368569 | 1.422101 | 1.477455 | 1.534687 | 1.593848 | 8 |
| 9 | 1.304773 | 1.362897 | 1.423312 | 1.486095 | 1.551328 | 1.619094 | 1.689479 | 9 |
| 10 | 1.343916 | 1.410599 | 1.480244 | 1.552969 | 1.628895 | 1.708144 | 1.790848 | 10 |
| 11 | 1.384234 | 1.459970 | 1.539454 | 1.622853 | 1.710339 | 1.802092 | 1.898299 | 11 |
| 12 | 1.425761 | 1.511069 | 1.601032 | 1.695881 | 1.795856 | 1.901207 | 2.012196 | 12 |
| 13 | 1.468534 | 1.563956 | 1.665074 | 1.772196 | 1.885649 | 2.005774 | 2.132928 | 13 |
| 14 | 1.512590 | 1.618695 | 1.731676 | 1.851945 | 1.979932 | 2.116091 | 2.260904 | 14 |
| 15 | 1.557967 | 1.675349 | 1.800944 | 1.935282 | 2.078928 | 2.232476 | 2.396558 | 15 |
| 16 | 1.604706 | 1.733986 | 1.872981 | 2.022370 | 2.182875 | 2.355263 | 2.540352 | 16 |
| 17 | 1.652848 | 1.794676 | 1.947900 | 2.113377 | 2.292018 | 2.484802 | 2.692773 | 17 |
| 18 | 1.702433 | 1.857489 | 2.025817 | 2.208479 | 2.406619 | 2.621466 | 2.854339 | 18 |
| 19 | 1.753506 | 1.922501 | 2.106849 | 2.307860 | 2.526950 | 2.765647 | 3.025599 | 19 |
| 20 | 1.806111 | 1.989789 | 2.191123 | 2.411714 | 2.653298 | 2.917757 | 3.207135 | 20 |
| 21 | 1.860295 | 2.059431 | 2.278768 | 2.520241 | 2.785963 | 3.078234 | 3.399564 | 21 |
| 22 | 1.916103 | 2.131512 | 2.369919 | 2.633652 | 2.925261 | 3.247537 | 3.603537 | 22 |
| 23 | 1.973587 | 2.206114 | 2.464716 | 2.752166 | 3.071524 | 3.426152 | 3.819750 | 23 |
| 24 | 2.032794 | 2.283328 | 2.563304 | 2.876014 | 3.225100 | 3.614590 | 4.048935 | 24 |
| 25 | 2.093778 | 2.363245 | 2.665836 | 3.005434 | 3.386355 | 3.813392 | 4.291871 | 25 |
| 26 | 2.156591 | 2.445959 | 2.772470 | 3.140679 | 3.555673 | 4.023129 | 4.549383 | 26 |
| 27 | 2.221289 | 2.531567 | 2.883369 | 3.282010 | 3.733456 | 4.244401 | 4.822346 | 27 |
| 28 | 2.287928 | 2.620172 | 2.998703 | 3.429700 | 3.920129 | 4.477843 | 5.111687 | 28 |
| 29 | 2.356566 | 2.711878 | 3.118651 | 3.584036 | 4.116136 | 4.724124 | 5.418388 | 29 |
| 30 | 2.427262 | 2.806794 | 3.243398 | 3.745318 | 4.321942 | 4.983951 | 5.743491 | 30 |
| 31 | 2.500080 | 2.905031 | 3.373133 | 3.913857 | 4.538039 | 5.258069 | 6.088101 | 31 |
| 32 | 2.575083 | 3.006708 | 3.508059 | 4.089981 | 4.764941 | 5.547262 | 6.453387 | 32 |
| 33 | 2.652335 | 3.111942 | 3.648381 | 4.274030 | 5.003189 | 5.852362 | 6.840590 | 33 |
| 34 | 2.731905 | 3.220860 | 3.794316 | 4.466362 | 5.253348 | 6.174242 | 7.251025 | 34 |
| 35 | 2.813862 | 3.333590 | 3.946089 | 4.667348 | 5.516015 | 6.513825 | 7.686087 | 35 |
| 36 | 2.898278 | 3.450266 | 4.103933 | 4.877378 | 5.791816 | 6.872085 | 8.147252 | 36 |
| 37 | 2.985227 | 3.571025 | 4.268090 | 5.096860 | 6.081407 | 7.250050 | 8.636087 | 37 |
| 38 | 3.074783 | 3.696011 | 4.438813 | 5.326219 | 6.385477 | 7.648803 | 9.154252 | 38 |
| 39 | 3.167027 | 3.825372 | 4.616366 | 5.565899 | 6.704751 | 8.069487 | 9.703507 | 39 |
| 40 | 3.262038 | 3.959260 | 4.801021 | 5.816365 | 7.039989 | 8.513309 | 10.285718 | 40 |

## TABLE II

Present value of 1 due at the end of $n$ periods

$1/(1+i)^n$

| $n$ | $\frac{3}{4}\%$ | 1% | $1\frac{1}{4}\%$ | $1\frac{1}{2}\%$ | $1\frac{3}{4}\%$ | 2% | $2\frac{1}{2}\%$ |
|---|---|---|---|---|---|---|---|
| 1 | 0.992556 | 0.990099 | 0.987654 | 0.985222 | 0.982801 | 0.980392 | 0.975610 |
| 2 | 0.985167 | 0.980296 | 0.975461 | 0.970662 | 0.965898 | 0.961169 | 0.951814 |
| 3 | 0.977833 | 0.970590 | 0.963418 | 0.956317 | 0.949285 | 0.942322 | 0.928599 |
| 4 | 0.970554 | 0.960980 | 0.951524 | 0.942184 | 0.932959 | 0.923845 | 0.905951 |
| 5 | 0.963329 | 0.951466 | 0.939777 | 0.928260 | 0.916913 | 0.905731 | 0.883854 |
| 6 | 0.956158 | 0.942045 | 0.928175 | 0.914542 | 0.901143 | 0.887971 | 0.862297 |
| 7 | 0.949040 | 0.932718 | 0.916716 | 0.901027 | 0.885644 | 0.870560 | 0.841265 |
| 8 | 0.941975 | 0.923483 | 0.905398 | 0.887711 | 0.870412 | 0.853490 | 0.820747 |
| 9 | 0.934963 | 0.914340 | 0.894221 | 0.874592 | 0.855441 | 0.836755 | 0.800728 |
| 10 | 0.928003 | 0.905287 | 0.883181 | 0.861667 | 0.840729 | 0.820348 | 0.781198 |
| 11 | 0.921095 | 0.896324 | 0.872277 | 0.848933 | 0.826269 | 0.804263 | 0.762145 |
| 12 | 0.914238 | 0.887449 | 0.861509 | 0.836387 | 0.812058 | 0.788493 | 0.743556 |
| 13 | 0.907432 | 0.878663 | 0.850873 | 0.824027 | 0.798091 | 0.773033 | 0.725420 |
| 14 | 0.900677 | 0.869963 | 0.840368 | 0.811849 | 0.784365 | 0.757875 | 0.707727 |
| 15 | 0.893973 | 0.861349 | 0.829993 | 0.799851 | 0.770875 | 0.743015 | 0.690466 |
| 16 | 0.887318 | 0.852821 | 0.819746 | 0.788031 | 0.757616 | 0.728446 | 0.673625 |
| 17 | 0.880712 | 0.844377 | 0.809626 | 0.776385 | 0.744586 | 0.714163 | 0.657195 |
| 18 | 0.874156 | 0.836017 | 0.799631 | 0.764912 | 0.731780 | 0.700159 | 0.641166 |
| 19 | 0.867649 | 0.827740 | 0.789759 | 0.753607 | 0.719194 | 0.686431 | 0.625528 |
| 20 | 0.861190 | 0.819544 | 0.780009 | 0.742470 | 0.706825 | 0.672971 | 0.610271 |
| 21 | 0.854779 | 0.811430 | 0.770379 | 0.731498 | 0.694668 | 0.659776 | 0.595386 |
| 22 | 0.848416 | 0.803396 | 0.760868 | 0.720688 | 0.682720 | 0.646839 | 0.580865 |
| 23 | 0.842100 | 0.795442 | 0.751475 | 0.710037 | 0.670978 | 0.634156 | 0.566697 |
| 24 | 0.835831 | 0.787566 | 0.742197 | 0.699544 | 0.659438 | 0.621721 | 0.552875 |
| 25 | 0.829609 | 0.779768 | 0.733034 | 0.689206 | 0.648096 | 0.609531 | 0.539391 |
| 26 | 0.823434 | 0.772048 | 0.723984 | 0.679021 | 0.636950 | 0.597579 | 0.526235 |
| 27 | 0.817304 | 0.764404 | 0.715046 | 0.668986 | 0.625995 | 0.585862 | 0.513400 |
| 28 | 0.811220 | 0.756836 | 0.706219 | 0.659099 | 0.615228 | 0.574375 | 0.500878 |
| 29 | 0.805181 | 0.749342 | 0.697500 | 0.649359 | 0.604647 | 0.563112 | 0.488661 |
| 30 | 0.799187 | 0.741923 | 0.688889 | 0.639762 | 0.594248 | 0.552071 | 0.476743 |
| 31 | 0.793238 | 0.734577 | 0.680384 | 0.630308 | 0.584027 | 0.541246 | 0.465115 |
| 32 | 0.787333 | 0.727304 | 0.671984 | 0.620993 | 0.573982 | 0.530633 | 0.453771 |
| 33 | 0.781472 | 0.720103 | 0.663688 | 0.611816 | 0.564111 | 0.520229 | 0.442703 |
| 34 | 0.775654 | 0.712973 | 0.655494 | 0.602774 | 0.554408 | 0.510028 | 0.431905 |
| 35 | 0.769880 | 0.705914 | 0.647402 | 0.593866 | 0.544873 | 0.500028 | 0.421371 |
| 36 | 0.764149 | 0.698925 | 0.639409 | 0.585090 | 0.535502 | 0.490223 | 0.411094 |
| 37 | 0.758461 | 0.692005 | 0.631515 | 0.576443 | 0.526292 | 0.480611 | 0.401067 |
| 38 | 0.752814 | 0.685153 | 0.623719 | 0.567924 | 0.517240 | 0.471187 | 0.391285 |
| 39 | 0.747210 | 0.678370 | 0.616018 | 0.559531 | 0.508344 | 0.461948 | 0.381741 |
| 40 | 0.741648 | 0.671653 | 0.608413 | 0.551262 | 0.499601 | 0.452890 | 0.372431 |

## TABLE II

Present value of 1 due at the end of $n$ periods

$1/(1+i)^n$

| $n$ | 3% | 3½% | 4% | 4½% | 5% | 5½% | 6% | $n$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.970874 | 0.966184 | 0.961538 | 0.956938 | 0.952381 | 0.947867 | 0.943396 | 1 |
| 2 | 0.942596 | 0.933511 | 0.924556 | 0.915730 | 0.907029 | 0.898452 | 0.889996 | 2 |
| 3 | 0.915142 | 0.901943 | 0.888996 | 0.876297 | 0.863838 | 0.851614 | 0.839619 | 3 |
| 4 | 0.888487 | 0.871442 | 0.854804 | 0.838561 | 0.822702 | 0.807217 | 0.792094 | 4 |
| 5 | 0.862609 | 0.841973 | 0.821927 | 0.802451 | 0.783526 | 0.765134 | 0.747258 | 5 |
| 6 | 0.837484 | 0.813501 | 0.790315 | 0.767896 | 0.746215 | 0.725246 | 0.704961 | 6 |
| 7 | 0.813092 | 0.785991 | 0.759918 | 0.734828 | 0.710681 | 0.687437 | 0.665057 | 7 |
| 8 | 0.789409 | 0.759412 | 0.730690 | 0.703185 | 0.676839 | 0.651599 | 0.627412 | 8 |
| 9 | 0.766417 | 0.733731 | 0.702587 | 0.672904 | 0.644609 | 0.617629 | 0.591898 | 9 |
| 10 | 0.744094 | 0.708919 | 0.675564 | 0.643928 | 0.613913 | 0.585431 | 0.558395 | 10 |
| 11 | 0.722421 | 0.684946 | 0.649581 | 0.616199 | 0.584679 | 0.554910 | 0.526788 | 11 |
| 12 | 0.701380 | 0.661783 | 0.624597 | 0.589664 | 0.556837 | 0.525982 | 0.496969 | 12 |
| 13 | 0.680951 | 0.639404 | 0.600574 | 0.564272 | 0.530321 | 0.498561 | 0.468839 | 13 |
| 14 | 0.661118 | 0.617782 | 0.577475 | 0.539973 | 0.505068 | 0.472569 | 0.442301 | 14 |
| 15 | 0.641862 | 0.596891 | 0.555264 | 0.516720 | 0.481017 | 0.447933 | 0.417265 | 15 |
| 16 | 0.623167 | 0.576706 | 0.533908 | 0.494469 | 0.458112 | 0.424581 | 0.393646 | 16 |
| 17 | 0.605016 | 0.557204 | 0.513373 | 0.473176 | 0.436297 | 0.402447 | 0.371364 | 17 |
| 18 | 0.587395 | 0.538361 | 0.493628 | 0.452800 | 0.415521 | 0.381466 | 0.350344 | 18 |
| 19 | 0.570286 | 0.520156 | 0.474642 | 0.433302 | 0.395734 | 0.361579 | 0.330513 | 19 |
| 20 | 0.553676 | 0.502566 | 0.456387 | 0.414643 | 0.376889 | 0.342729 | 0.311805 | 20 |
| 21 | 0.537549 | 0.485571 | 0.438834 | 0.396787 | 0.358942 | 0.324862 | 0.294155 | 21 |
| 22 | 0.521892 | 0.469151 | 0.421955 | 0.379701 | 0.341850 | 0.307926 | 0.277505 | 22 |
| 23 | 0.506692 | 0.453286 | 0.405726 | 0.363350 | 0.325571 | 0.291873 | 0.261797 | 23 |
| 24 | 0.491934 | 0.437957 | 0.390121 | 0.347703 | 0.310068 | 0.276657 | 0.246979 | 24 |
| 25 | 0.477606 | 0.423147 | 0.375117 | 0.332731 | 0.295303 | 0.262234 | 0.232999 | 25 |
| 26 | 0.463695 | 0.408838 | 0.360689 | 0.318402 | 0.281241 | 0.248563 | 0.219810 | 26 |
| 27 | 0.450189 | 0.395012 | 0.346817 | 0.304691 | 0.267848 | 0.235604 | 0.207368 | 27 |
| 28 | 0.437077 | 0.381654 | 0.333477 | 0.291571 | 0.255094 | 0.223322 | 0.195630 | 28 |
| 29 | 0.424346 | 0.368748 | 0.320651 | 0.279015 | 0.242946 | 0.211679 | 0.184557 | 29 |
| 30 | 0.411987 | 0.356278 | 0.308319 | 0.267000 | 0.231377 | 0.200644 | 0.174110 | 30 |
| 31 | 0.399987 | 0.344230 | 0.296460 | 0.255502 | 0.220359 | 0.190184 | 0.164255 | 31 |
| 32 | 0.388337 | 0.332590 | 0.285058 | 0.244500 | 0.209866 | 0.180269 | 0.154957 | 32 |
| 33 | 0.377026 | 0.321343 | 0.274094 | 0.233971 | 0.199873 | 0.170871 | 0.146186 | 33 |
| 34 | 0.366045 | 0.310476 | 0.263552 | 0.223896 | 0.190355 | 0.161963 | 0.137912 | 34 |
| 35 | 0.355383 | 0.299977 | 0.253415 | 0.214254 | 0.181290 | 0.153520 | 0.130105 | 35 |
| 36 | 0.345032 | 0.289833 | 0.243669 | 0.205028 | 0.172657 | 0.145516 | 0.122741 | 36 |
| 37 | 0.334983 | 0.280032 | 0.234297 | 0.196199 | 0.164436 | 0.137930 | 0.115793 | 37 |
| 38 | 0.325226 | 0.270562 | 0.225285 | 0.187750 | 0.156605 | 0.130739 | 0.109239 | 38 |
| 39 | 0.315754 | 0.261412 | 0.216621 | 0.179665 | 0.149148 | 0.123924 | 0.103056 | 39 |
| 40 | 0.306557 | 0.252572 | 0.208289 | 0.171929 | 0.142046 | 0.117463 | 0.097222 | 40 |

### Review Set No. 1. Pages 8-9

1. 3.
2. $-5$.
3. 35.
4. 15.
5. 1.
6. 24.
7. 3.
8. $\frac{7}{3}$.
9. 0.
10. $-5$.
11. $-9.4$.
12. $-\frac{1}{4}$.
13. 13.
14. $11ab$.
15. $5m^2$.
16. $-7xy$.
17. $2x$.
18. $-3.6a^2b^2$.
19. $-2x$.
20. $-0.7pq$.
21. $\frac{9}{20}lw$.
22. $-17$.
23. 1.2.
24. $-2x$.
25. 100.
26. $-12y^2z$.
27. $-5x^2$.
28. $5a$.
29. $-\frac{1}{4}x^2$.
30. $-2.1ab$.
31. $-a^2-8b^2$.
32. $2x+9y$.
33. $x^2-10x-4$.
34. $-3a^2+8ab-11b^2$.
35. $3y$.
36. $-7mn+kn-5km$.
37. $x^2-3$.
38. $-3y$.
39. $9r^3-8r^2-8s^3$.

### Review Set No. 2. Pages 9-11

1. 61.
2. 0.
3. $\frac{1}{3}$.
4. 9.
5. $-\frac{27}{20}$.
6. $-5x^2$.
7. $-1.8ab$.
8. $4z$.
9. $-29$.
10. $-a^2b-11ab^2$.
11. $-7x^2+xy+9y^2$.
12. $m^3-8m^2+8m-5$.
13. $-x^2-2x+10$.
16. $2(x-4y)$.
17. $a(x+y)$.
18. $3x(x+2)$.
19. $ay(y+1)$.
20. $ab(ab-3)$.
21. $2x(2x-3)$.
22. $4xy(3x^2y^2-1)$.
23. $3r^2(r+4)$.
24. $2mn(3m+n-2mn)$.
25. $12a^2$.
26. $-12a^2$.
27. $-12a^2$.
28. $12a^2$.
29. $28m^3n^3$.
30. $2b^4$.
31. $-8.6x^3$.
32. $a^3-a$.
33. $-6x^4-12x^2y$.
34. $18a^2-6ab$.
35. $-12x^2-8xy$.
36. $10m^2+15mn$.
37. 1.
38. 12.
39. $x$.
40. $-x$.
41. $2m^2$.
42. $-x^2$.
43. $x$.
44. $-2a$.
45. $-3y^3$.
46. $a+1$.
47. $2x-3$.
48. $-2rs+1$.
49. $14a+6$.
50. $4x^2+2ax+3a^2$.
51. $x^2-18xy$.
52. $2a^2-12a+9$.

**Review Set No. 3. Pages 11-13**

1. 12.
2. 5, 12, 9, $7\frac{3}{4}$, 4.07.
3. $4.9a$.
4. $-5x^2$.
5. $-8.5xy$.
6. $6m^2n^3$.
7. $-14y^2$.
8. $5(x+y)$.
9. $-7(a-b)$.
10. $-2.4a^2$.
11. $-(m+n)$.
12. $11ab$.
13. $3x$.
14. $-22x^2y$.
15. $-11x$.
16. $37xyz$.
17. $3(x+y)$.
18. $-6.2t$.
19. $-16a^3b^3$.
20. $-7m^2n$.
21. $-12x^2$.
22. $16xy^2$.
23. $30x^3$.
24. $-\frac{3}{8}a^3b^3$.
25. $1.2a^3b^3$.
26. $-24x^3y^3$.
27. $-8r^2+12rs$.
28. $2a^2+3ab-2b^2$.
29. $3x+2xy-8y^2$.
30. $-x$.
31. $-5x$.
32. $3b$.
33. $-3k$.
34. $-\frac{1}{2}a^2b^2$.
35. $-4q$.
36. 7.
37. $x-2y$.
38. $2a-5b$.
39. $-x-2$.
40. $m-1$.
41. $2r+5t$.
42. $\frac{5}{7}$.
43. $5/x$.
44. $\frac{3}{20}$.
45. $3/2a$.
46. $\frac{5}{18}$.
47. $\dfrac{c+a}{abc}$.
48. $7/12a$.
49. $(5+x)/x^2$.
50. $\frac{1}{48}$.
51. $-3b$.
52. 1.
53. $5kmn$.
54. $\dfrac{-5yz^5}{2w}$.
55. $9x$.
56. $3k-5m$.

**Review Set No. 4. Pages 13-15**

1. $V=1540$.
2. $A=24$.
3. $s=28.5$.
4. $C=100$.
5. $S=5050$.
6. $D=10.25$.
7. $V=4\pi r^3/3$; $r$ feet, $V$ cubic feet, etc.
8. $d=rt$; $d$ miles, $r$ miles per hour, $h$ hours, etc.
9. $i=prt$; $\$i$, $\$p$, $r$ is rate per annum, $t$ years, etc.
10. $c=2\pi r$; $c$ inches, $r$ inches, etc.
11. $A=cl$; $A$ sq. cm., $c$ cm., $l$ cm., etc.
12. $y=3x$.
13. $c=2.5i$.
14. $f=s+24$.
15. $h=V/\pi r^2$; $r=\sqrt{V/\pi h}$.
16. $h=2A/(a+b)$.
17. $F=\frac{9}{5}C+32$.
18. $n=2S/(a+l)$.
19. $V=k/P$.
20. $S=(R-b)/a$.

**21.** (a) $V$ is increased 4-fold; (b) $V$ is halved; (c) $V$ is increased 8-fold.
**22.** (a) $r$ is doubled; (b) $r$ is halved; (c) $r$ is increased 4-fold.
**23.** (a) $V$ is increased 8-fold; (b) $V$ is reduced to $\frac{1}{8}$th of its former value; (c) $V$ is increased 27-fold.
**24.** $V$ is (a) 4 times as large, (b) 6 times as large, (c) the same.
**25.** $a^2(4-\pi)$; $2a(2+\pi)$.
**26.** $\pi r^2/2$; $2\pi r$.
**27.** $m^2(4+2\pi)$; $4\pi m$.
**28.** $x^2(6-\frac{5}{4}\pi)$; $3\pi x$.

### Review Set No. 5. Pages 15-16

**1.** 2.
**2.** 12.
**3.** 7/5.
**4.** 35.
**5.** $-4$.
**6.** $-2$.
**7.** $-3/8$.
**8.** $-6$.
**9.** 2.
**10.** 3.
**11.** $-2$.
**12.** 15.
**13.** 3.
**14.** $-5$.
**15.** $-8$.
**16.** 0.3.
**17.** $b/a$.
**18.** $b-a$.
**19.** 5.
**20.** 1.
**21.** 1/2.
**22.** 4/3.
**23.** 3.1.
**24.** $-2$.
**25.** 3.
**26.** 4.
**27.** 8.
**28.** 8.
**29.** 1.
**30.** 6.
**31.** 3/2.
**32.** $-2$.
**33.** 10.
**34.** 10.
**35.** 10.
**36.** 14.
**37.** 1.
**38.** 24.
**39.** $-3$.
**40.** $-1/2$.
**41.** $-1/2$.
**42.** 2.5.
**43.** 7.
**44.** 24.
**45.** $-1$.
**46.** $-2$.

### Review Set No. 6. Pages 16-19

**1.** $x+6$.
**2.** $11+x$.
**3.** $x-4$.
**4.** $2x$.
**5.** $x/2$.
**6.** $3x+15$.
**7.** $3(x+15)$.
**8.** $x^3$.
**9.** $7x^2$.
**10.** $\frac{1}{4}x+9$.
**11.** $5x=x+8$.
**12.** $5+x=12$.
**13.** $x+6=2x$.
**14.** $3x^2=75$.
**15.** $2x-5=3x+7$.
**16.** 15.
**17.** 15.
**18.** 12, 60.
**19.** 8 feet.
**20.** 28.
**21.** 24 rods by 48 rods.
**22.** \$60, \$120, \$180.
**23.** 9 years.
**24.** \$4.96.
**25.** \$35.
**26.** \$2030.00.
**27.** 175 feet.

28. 42°, 62°, 76°. 29. 28°, 84°, 68°. 30. 29 and 30.
31. 200 at 15c.; 300 at 25c. 32. 42, 43, 44. 33. 4.5 miles.
34. 8 ft., 18 ft., 18 ft.

### Page 24

1. $x=7$, $y=4$. 2. $x=10$, $y=3$. 3. $x=3$, $y=2$.
4. $x=-5$, $y=1$. 5. $x=2$, $y=-1$. 6. $x=6$, $y=-1$.
7. $x=1$, $y=4$. 8. $x=1$, $y=\frac{1}{2}$. 9. $x=3$, $y=-2$.
10. $x=2$, $y=3$.

### Page 26

4. $x=1$, $y=\frac{1}{2}$. 5. $x=3$, $y=2$. 6. $x=1$, $y=-1$.
7. $x=8$, $y=2$. 8. $x=6$, $y=5$. 9. $x=0.3$, $y=-0.2$.
10. $x=\frac{1}{2}$, $y=-\frac{1}{2}$. 11. $x=2$, $y=-1$. 12. $x=4$, $y=-1$.
13. $x=0.4$, $y=-0.2$. 14. $x=1$, $y=2$. 15. $x=\frac{5}{4}$, $y=\frac{5}{3}$.
16. $x=\frac{1}{2}$, $y=\frac{1}{5}$. 17. $x=-2$, $y=3$.

### Page 27

1. $x=7$, $y=4$. 2. $x=10$, $y=3$. 3. $x=-5$, $y=1$.
4. $x=2$, $y=-1$. 5. $x=1$, $y=4$. 6. $x=1$, $y=\frac{1}{2}$.
7. $x=3$, $y=-2$. 8. $x=2$, $y=3$.

### Pages 29-31

1. 17, 20. 2. 5, 4. 3. $6\frac{3}{7}$, $\frac{3}{7}$.
4. 10, 15. 5. 35 rods, 25 rods. 6. 80 lb. at 40c.; 20 lb. at 80c.
7. 8 feet, 24 feet. 8. John gets \$135, Mary \$115.
9. 75, 100. 10. 100 at 3c.; 50 at 4c.
11. 30 miles. 12. 650 lb. at 30c.; 350 lb. at 40c.
13. 105 nickels, 35 dimes. 14. 5.

### Pages 32-33

1. $y$ increases; $x$ increases; yes.
2. $y$ increases; $y$ decreases; no; yes.
3. $y$ decreases; $x$ decreases; $y$ decreases by 3; $y$ decreases by 6.

### Page 35

3. (a) $x$ and $y$ both positive; (b) $x$ negative, $y$ positive; (c) $x$ and $y$ both negative; (d) $x$ positive, $y$ negative; (e) $y=0$; (f) $x=0$.
4. (0,0), (0,5), (5,5), (5,0). 5. $(-5,4)$.
6. (a) (4,7); (b) (1,3); (c) $(-3,-8)$; (d) (0,0).
7. (0,0), $(-6,0)$, $(-6,-4)$, $(0,-4)$; $(-3,-2)$.

### Pages 40-41

1. A line parallel to the $y$-axis and 3 units to the right.
3. The $y$-axis. 5. Line parallel $x$-axis, 5 units below.
7. Line parallel $y$-axis, 3/2 units to right.
9. Line parallel $y$-axis, 8 units to left.
11. Line bisecting angle between axes, in first and third quadrants.

19. (a) 8; (b) $-6$. 20. (a) $y$ increases; (b) $x$ increases by 2.
21. (a) $x$ decreases; (b) a decrease of 4.
22. (a) The point $(x,y)$ moves so that its $x$-coordinate equals twice its $y$-coordinate.
(b) The point $(x,y)$ moves so that its $x$-coordinate is 3 less than three times its $y$-coordinate.

### Page 44

1. (1,1). 2. (2,1). 3. (0,0).
4. $(3,-2)$. 5. No solution. 6. (3,2).
7. (3,0). 8. Unlimited number of solutions.

### Review Exercises on Chapter II. Pages 44-46

2. $x=-3$, $y=-1$. 3. 40 and 52. 6. (0,0),(3,0), (3,3),(0,3).
7. $(2,-2)$. 9. $y$ increases; 4 units.
11. One, none, or an unlimited number.
12. First, fourth, third, second; negative $x$-axis, positive $y$-axis.
13. See page 34. 14. An increase of 1 unit.
15. 45, 25. 16. (1,3). 18. 5/12.

19. $m=2$, $n=-1$.
20. $a=6$, $b=4$.
21. $x=0.5$, $y=-1.2$.
22. $x=4$, $y=0$.
24. \$550 at 2%; \$250 at 3%.
25. 15″ by 18″.
26. 30′ by 66′.
27. (1) $y$ decreases; (2) $x$ decreases; (3) $y$ decreases by 1 unit; (4) the $x$-coordinate plus 5 times the $y$-coordinate equals 5.
28. $x=4$, $y=3$.

**Pages 49-50**

1. $A$ is increased 4-fold; 16-fold; $r$ is diminished to $\frac{1}{3}$ of its former value; $r=\sqrt{A/\pi}$.
2. 100; between 0° and 32° $F$; $F=\frac{9}{5}C+32$.
3. $F$ is decreased to $\frac{1}{4}$ of its former value; $F$ is increased 4-fold; decrease $r$ to $\frac{1}{5}$ of its value; $F$ decreases from 100 to 1; $r=10/\sqrt{F}$.
4. $l=V/wh$ etc; reduce $h$ to $\frac{1}{4}$ of its value; $V$ is decreased by 25%.
5. (a) $d$ is doubled; (b) $t$ is halved; (c) increase of 50% in $t$; (d) increase of 100% in $t$.

**Pages 53-54**

1. (b) (i) $p$ increases by 12, (ii) $p$ increases by $3k$, (iii) $p$ increases 3-fold, (iv) $p$ increases $k$-fold.
4. $F=32$ when $C=0$; $C=-17\frac{7}{9}$ when $F=0$.

**Pages 69-71**

1. (a) 90°, (b) 45°, (c) 180°.
2. (a) 15 min., (b) 3 hours.
3. (a) 720, (b) 60, (c) 45, (d) 30, (e) 1.
4. 15.
5. (a) 30, (b) 180, (c) 120, (d) 60, (e) 150, (f) 30, (g) 60, (h) 120.
6. (a) $\frac{1}{3}$, (b) $\frac{1}{24}$, (c) $\frac{1}{16}$, (d) $\frac{1}{8}$, (e) $\frac{1}{6}$, (f) $\frac{1}{5}$, (g) $\frac{3}{10}$, (h) $\frac{3}{8}$, (i) $\frac{5}{8}$, (j) $\frac{2}{3}$, (k) $\frac{3}{4}$, (l) $\frac{5}{6}$.
7. (a) 30, (b) 360.
8. (a) 90°, (c) 45°, (e) 45°, (g) 90°, (i) 135°.

### Page 82

4. 60°, 30°.

### Page 84

12. 18. 13. 120°. 14. 45°, 135°.
15. 35°, 55°. 16. 100°, 80°. 17. 30°, 150°.
18. 45°.

### Pages 107-109

4. (b) 4, 2. 5. $x=22$; $\angle RQS=110°$.
12. (f) 2; 1, 1, 3, 3, 2, 2.

### Page 139

3. 120°. 4. 120°.
5. (a) 50°, (b) 130°, (c) 50°, (d) 130°.

### Page 149

3. (a) If $x$ is unequal to $p$, then $x-p$ is not equal to zero.
(e) If the liquid does not contain an acid, then it does not turn blue litmus red.

### Page 155

1. (a) 25°, (b) 155°. 2. (a) 60°, (b) 120°. 3. (a) 80°, (b) 100°.

### Page 163

11. $\angle A=36°$; $\angle B=\angle C=72°$.
12. $\angle A=90°$; $\angle B=30°$; $\angle C=60°$.
13. 40°, 80°, 60°. 15. $22\frac{1}{2}°$, $67\frac{1}{2}°$.

### Page 178

4. (a) 60°, (b) $\angle A=40°$, $\angle C=80°$.

### Pages 183-184

2. (a) 18 cm., (b) 20 sq. cm. 4. 4 cm. 6. 40 sq. cm.
8. (a) 961; (b) $49n^2$; (c) $6\frac{1}{4}p^2$; (d) $0.25r^2$; (e) $0.0004s^2$;
(f) $x^2+2x+1$; (g) $a^2+2ab+b^2$; (h) $a^2-10a+25$;

(i) $x^2+14x+49$; (j) $x^2-22x+121$.

**10.** (a) $6ab$; (b) $20a^2$; (c) $4a+8$; (d) $6m+18$; (e) $a^2+5a$; (f) $m^2-7m$; (g) $3k^2+9k$; (h) $a^2-4$; (i) $x^2-y^2$; (j) $x^2+9x+20$; (k) $m^2+16m+64$; (l) $p^2+13p+42$.

**12.** (a) $10xy$; (b) $12x^2$; (c) $a^2-ab$; (d) $8x^2-24x$; (e) $2a^2+3ab-2b^2$; (f) $3x^2+2xy-8y^2$.

### Pages 187-193

**2.** (a) 4 in., (b) 2 in.

**4.** (a) $2a$, (b) $x$, (c) $2c+10$, (d) $2d$, (e) $2y$.

**6.** (a) 4, (b) 12.

**20.** 3.7 in. (correct to 2 figs.)

**22.** 2 units.

**28.** 60 sq. cm.

**29.** 96 sq. rods.

**33.** (a) 65 sq. cm.; (b) 15 sq. cm.

**35.** (a) 2 $(p_1-p_2)$, (b) $(p_1+p_2)^2$, (c) $\frac{1}{2}d^2$.

**36.** 7200 sq. ft.

**37.** About 91.

### Page 195

**1.** (a) $\frac{1}{20}$; (b) $\frac{20}{1}$; (c) $\frac{1}{6}$; (d) $\frac{6}{1}$; (e) $\frac{11}{12}$; (f) $\frac{16}{1}$; (g) $\frac{1}{16}$; (h) $\frac{1}{10}$; (i) $\frac{7}{8}$; (j) $\frac{1}{4}$; (k) $\frac{1}{3}$.

### Pages 196-197

**5.** $2\frac{1}{2}$.

**7.** 8.

**9.** (a) $\frac{1}{2}$, (c) 7, (e) $\frac{q}{p}$, (f) $\frac{a}{m+n}$, (h) 11.

### Page 202

**7.** 10 units.

**8.** 12 units.

**9.** 4 ft.

### Pages 207-209

**4.** (a) $\frac{2}{3}$, (b) 1.6 cm.

**5.** (a) $\frac{3}{4}$, (b) 9 units.

**6.** (a) 6, (b) 3.

**10.** 8.4 in.

### Pages 214-217

**2.** 24 ft. **6.** 24 ft. **8.** 195 sq. in. **12.** (a) $\sqrt{\frac{2}{1}}$.
**14.** $DE = 7.1$ ft., $DF = 7.1$ ft., $AD = 10$ ft., $AE = 7.1$ ft., $AF = 7.1$ ft.

### Review Exercises. Pages 248-261

**12.** 45°.
**14.** 75°, 15°.
**16.** 90°.
**19.** (a) $\frac{5}{2}$, (b) $5\frac{5}{16}$ in., (c) $3\frac{3}{4}$ in.
**31.** $x^2+11x+24$ square units.
**32.** $m^2-2mn+n^2$ square units.
**33.** $x+8$ units.
**36.** $x-5$ units.
**37.** $\frac{d}{2}(x+y)$ square units.
**39.** $s^2$ square units. A square.
**41.** $x^2+13x+42$ square units.
**43.** $2\frac{2}{3}$ hours.
**95.** 80°.
**98.** The areas are equal.
**110.** 2.8, to 2 figs.
**113.** 84 cm.

### Pages 271-272

**1.** 52 sq. ft.
**2.** (a) 35,200 cu. in., (b) 6080 sq. in.
**3.** 9.1 tons.
**4.** 12 gal.
**5.** 92 tons.
**6.** 3800 cu. in.
**7.** 230 sq. ft.
**8.** 150 ft.
**9.** 384,000 gal.
**10.** 7.44 ft.

### Page 274

**1.** (a) 1/6, (b) 1/8, (c) 1/36, (d) 1/3, (e) 1/2, (f) 1/5.
**2.** (a) 1/4, (b) 2/7, (c) 1/10.
**3.** 6.42 sq. cm.
**4.** 24.5 sq. ft.
**5.** (a) 9.9 sq. in., (b) 9.0 sq. in.
**6.** (a) 120 sq. ft., (b) 539 sq. ft.
**7.** 36°; 6 cm.
**8.** (a) $22\frac{1}{2}$°, (b) 3 in.

### Page 276

**1.** 6 sq. in.
**2.** 30 sq. ft.
**3.** 84 sq. yd.
**4.** 96 sq. cm.
**5.** 8; 6.
**6.** 4 in., 3 in., 2.4 in.
**7.** 13 yd., 12 yd., 11 yd.

### Pages 282-283

1. 42 cu. in.
2. 100 cu. in.
5. 270 c.c., to 2 figs.
6. (a) $h = 3V/b$; (b) $b = 3V/h$.
7. 4.5 in.
8. 400 sq. cm.
9. 2500 cu. in., to 3 figs.
10. (a) 3450 cu. ft. to 3 figs.; (b) 1000 sq. ft. to 3 figs.
11. 51,900 cu. ft.
12. 13,800 cu. ft. to 3 figs.
13. (a) 480 cu. ft.; (b) 164 sq. ft; (c) 5 ft.; (d) 120 sq. ft.

### Pages 289-290

1. (a) $1437\frac{1}{3}$ c. in.; (b) 616 sq. in.
2. 1540 c.c.
3. (a) $r = \sqrt{3V/\pi h}$; (b) $h = 3V/\pi r^2$.
4. (a) an 8-fold increase; (b) a 4-fold increase.
5. About 64 times as great. About 16 times as great.
6. 800 c. ft.
7. (a) $20 \times 10^7$ sq. mi.; (b) $27 \times 10^{10}$ c. mi.
8. 75.4 sq. ft.
9. 1440 c. in., to 3 figs.
10. About 60.
11. 528 sq. units.
12. $14.90.
13. 4850 c. ft.; 30,200 gals.
14. 283 sq. ft.

### Review Exercises. Pages 290-294

1. 1300 sq. rods.
2. 1/2 rod; 50 sq. rods.
3. 1300 c.c.; 50 c.c.
4. 238 cu. ft.
5. 4440 c.c.
6. 72 cu. ft.
7. 750 c.c.
8. 4020 c.c.
9. 25,800 cu. in.
10. 84 sq. in.; 120 sq. in.
11. 24 in., 10 in., 9.2 in.
12. (a) 2.46 sq. ft.; (b) 0.84 sq. ft.
13. 33.7 sq. ft.
14. $31.
15. 1/4.
16. 8.4 cu. ft.
17. 16,300 c.c.
18. 790 cu. in.
19. About 20 ft.
20. Four times as large; no.

**21.** $44.
**22.** 50 ft.; 3470 sq. ft. (to 3 figs.).
**23.** 1500 sq. ft.
**24.** 31 ft.
**25.** 9680 cu. ft., to 3 figs.
**26.** (a) 43,400 cu. in.; (b) 6880 sq. in.
**27.** 105,000 cu. in.; 11,900 sq. in.
**28.** Volume becomes (a) 8 times as great, (b) 1/8th as great, (c) 27 times as great, (d) 27/8ths as great.
**29.** $125\pi/48$ cu. in.; 58%; 42%.
**30.** 51,000 gal.
**31.** $98.
**32.** 10.6 ft.
**33.** 340.
**34.** The area becomes (a) 9 times as great, (b) $\frac{1}{4}$ as great, (c) 9/16ths as great.
**35.** The volume becomes (a) 4 times as great, (b) twice as great, (c) $\frac{1}{2}$ as great, (d) 9/8ths as great.
**36.** 41,000,000.

### Pages 298-299

**1.** $10.
**2.** $5.50.
**3.** $14.
**4.** 13.6% per annum.
**5.** 18% approx.
**6.** (a) $7; (b) 25%.
**7.** (a) $33; (b) 18% approx.
**8.** (a) $25; (b) 14.7% approx.
**9.** 16% approx.

### Page 301

**1.** $80.88.
**2.** $1545.88.

### Page 303

**1.** $19.58
**2.** $23.07.
**3.** $6.15.
**4.** $18.14.
**5.** $4.60.
**6.** $4.06.
**7.** $103.23.
**8.** $9.53.

### Pages 306-307

**1.** $\$50(1.05)^{10}$.
**2.** $\$100(1.03)^{15}$.
**3.** $\$1200(1.035)^{5}$.
**4.** $\$260(1.02)^{16}$.
**5.** $\$130(1.015)^{14}$.
**6.** $\$750(1.015)^{17}$.
**7.** $\$3600(1.025)^{21}$.
**8.** $\$165(1.0125)^{9}$.
**9.** $115.76.

**10.** \$292.46. **11.** \$627.41. **12.** \$206.08.
**13.** \$112.93. **14.** \$9.42. **15.** $p\{(1+r)^n-1\}$.
**16.** $2p(1.025)^{2n}$.

### Pages 310-311

**1.** $\$200/(1.03)^2$. **2.** $\$350/(1.04)^8$. **3.** $\$185/(1.035)^{10}$.
**4.** $\$820/(1.02)^{24}$. **5.** $\$1000/(1.015)^{30}$. **6.** $\$425/(1.0125)^{14}$.
**7.** $\$105/(1.025)^{13}$. **8.** $\$640/(1.0175)^9$. **9.** \$100.
**10.** \$188.46. **11.** \$889.00. **12.** \$883.85.
**13.** \$27.88. **14.** $D/(1.02)^{27}$.

### Page 312

**1.** \$1168.48. **2.** \$2737.17. **3.** \$7669.98.
**4.** \$870.78. **5.** \$1608.25. **6.** \$8735.79.
**7.** (a) \$62.89; (b) \$63.86; (c) \$48.59; (d) \$48.89.
**8.** (a) \$92.46; (b) \$92.38; (c) \$744.09; (d) \$456.39.

### Pages 318-320

**1.** \$128.75. **2.** \$118.45. **3.** $7\frac{1}{2}\%$.
**4.** \$142.15. **5.** \$944.
**6.** \$17.30 per \$1000; less; more. **8.** \$306.87 remains.
**9.** (a) \$1196.40; (b) \$1372; (c) \$175.60. **10.** \$516.

### Exercises on Chapter XIV. Pages 320-325

#### A.

**1.** \$0.01.
**2.** (a) \$12.66, \$262.66; (b) \$6.95, \$81.95.
**3.** (a) \$4.55, \$105.55; (b) \$21.01, \$481.01.
**4.** \$0.58.
**5.** \$74.41.
**6.** (a) \$326.80, \$13.20; (b) \$103.66, \$16.34.
**7.** (a) \$604.76; (b) \$855.36.
**8.** (a) \$551.12; (b) \$298.99.
**9.** (a) \$607.53; (b) \$276.84.
**10.** \$8.50.
**11.** \$28.50.
**12.** \$9789.72.

B.

16. $119.11; $120.80; $1.69.
17. (a) 35 years; (b) between 20 and 21 years.
18. 2% per annum.
20. $725.42; $820.75.
21. 13% approx.
22. (a) 10%; (b) 11% approx.
23. (a) 7.1%; (b) 6.7%.
24. (b) 17%.
25. (a) $17.21; (b) 6.2%.
26. 6.2%.
28. $971.00; $189.94.
29. $16.78.
30. $524.40.

C.

31. 1.080000; 1.166400; 1.259712; 1.360489; 1.469328.
32. 0.925926; 0.857339; 0.793832; 0.735030; 0.680583.
33. Between 37 and 38 years.
34. $687.47.
35. $334.55.
36. $7.91.

**Page 330**

2. 3/5; 3; 1; 1/2.

**Pages 333-335**

1. 2.
2. 2/5, 5/2.
3. 2, 1/2.
4. 45 ft. nearly.
5. 36 ft.
6. 30 ft.
7. 230 ft. correct to 2 figs.
8. 202.5 ft.
9. 220 ft. correct to 2 figs.
10. (a) 60 ft.; (b) 247 ft.

**Pages 336-338**

1. 65 ft. (to 2 figs.).
3. 120 ft. (to 2 figs.).
4. 370 ft.
5. 2200 yards (to 2 figs.).
6. 130 ft. (to 2 figs.).
7. 56 ft.
8. 15°.
9. 21 ft.
10. 790 ft. (to 2 figs.) north from the house.

**Pages 343-344**

3. 13 ft.; 10.5 ft.
4. 3.5 in. approx.
5. 8.5 in.
6. 10 in.; 8.7 in.

7. 4.6 in.; 9.2 in.
8. 28.3 ft.
9. (a) 60°; (b) 16 ft.
10. 4.7 mi.; 11 mi.
11. (a) 24 ft.; (b) 19 ft.
12. (a) 1.9 ft.; (b) 1.7 ft.
13. (a) 6.9 in.; (b) 19 in.
14. (a) 30°; (b) 60°.

**Pages 344-348**

**A.**

1. 3.7 in.; 3.7 in.; 4.6 in.
3. (a) 0.1; (b) 0.1, 6°.
4. 138°.
5. 85 ft.
6. 2.1 in.
7. 31°; 59°.
8. 28 ft.
9. 500 ft. (to 2 figs.).
10. 35 ft.
11. 160 ft. (to 2 figs.).
12. (a) 65° approx.; (b) 33 ft.
13. 340 ft. (to 2 figs.).
14. 18°.
15. 9000 ft.

**B.**

16. $\sin B = 4/5$, $\cos B = 3/5$, $\tan B = 4/3$.
17. 34°.
18. 950 ft. (to 2 figs.).
19. 4 ft.
20. $5\frac{1}{4}$ ft.
21. 230 ft. (to 2 figs.).
22. $9\frac{1}{2}$ ft.
23. 27°.
24. 0.03; 0.4.
25. 1900 ft. (to 2 figs.).
26. 27°.

**C.**

27. 2500 ft. (assuming 2-figure accuracy).
28. 35 ft.
29. 73 ft.
30. 3.3°.
31. 42°; 19°.
32. 9.8 ft.
33. 220 ft. (to 2 figs.).

**Page 351**

1. (a) $x=0$ or $y=0$, (b) $x=0$ or $y=0$ or $z=0$, (c) $x=0$ or $y=3$, (d) $x=2$ or $y=0$, (e) $x=2$ or 3.
2. (a) $x=7$ or 5, (b) $x=7$ or $-5$, (c) $x=-7$ or $-5$.

**Page 352**

1. 3 in. or 5 in. from edge.

**Page 355**

1. $(x+5)(x+2)$.
2. $(x-5)(x-2)$.
3. $(x+5)(x-2)$.
4. $(x-5)(x+2)$.
5. $(a+5)(a+10)$.
6. $(a-5)(a-10)$.

7. $(x+1)(x+3)$.
8. $(x-1)(x-3)$.
9. $(x-3)(x+1)$.
10. $(x+3)(x-1)$.
11. $(m+6)^2$.
12. $(m-6)^2$.
13. $(x-3)(x-15)$.
14. $(x+5)(x-4)$.
15. $(a-5)(a+4)$.
16. $(k+5)(k+7)$.
17. $(x-2)(x-9)$.
18. $(x+1)^2$.
19. $(n-1)(n-2)$.
20. $(x-7)(x+4)$.
21. $(a+7)(a-4)$.
22. $(x+5)(x+25)$.
23. $(b-3)(b-30)$.
24. $(a-4)(a+3)$.
25. $(x+4)(x-3)$.
26. $(x-2)(x-25)$.
27. $(m-6)(m+5)$.
28. $(m+6)(m-5)$.
29. $(x+6)(x+7)$.
30. $(x-3)(x-15)$.
31. $(x-8)(x+7)$.
32. $(k+8)(k-7)$.
33. $(x+8)(x-6)$.
34. $(x-9)(x+4)$.

**Page 356**

1. $x^2+11x+28$.
2. $x^2-11x+28$.
3. $x^2-3x-28$.
4. $x^2+3x-28$.
5. $2x^2+7x+3$.
6. $3x^2+17x+10$.
7. $2x^2+5x-3$.
8. $3x^2+13x-10$.
9. $4x^2+15x+9$.
10. $4x^2-17x-15$.
11. $10x^2-3x-4$.
12. $10x^2-13x+4$.
13. $6x^2+13x+6$.
14. $6x^2-13x+6$.
15. $3x^2-10x-25$.
16. $3x^2-20x+25$.
17. $4x^2-24x+36$.
18. $15x^2+14x-8$.
19. $21x^2+10x+1$.
20. $21x^2+4x-1$.
21. $21x^2-10x+1$.
22. $21x^2-4x-1$.
23. $10y^2-11y-6$.
24. $15a^2-7a-4$.
25. $9b^2-13b-10$.
26. $16x^2+2x-3$.
27. $4m^2-25$.
28. $12k^2+8k-15$.
29. $15x^2-4x-32$.
30. $35x^2-8x-16$.

**Pages 357-358**

1. $(2x+3)(x+1)$.
2. $(2x-1)(x+3)$.
3. $(2x+5)(x+1)$.
4. $(2x-5)(x+1)$.
5. $(3x+2)(x+3)$.
6. $(3x-2)(x+3)$.
7. $(3x-2)(x-3)$.
8. $(3x+2)(x-3)$.
9. $(2x+3)(x-2)$.
10. $(2x+3)(x+2)$.
11. $(2x+4)(x+5)$.
12. $(2x-5)(x+4)$.
13. $(3x-4)(x-3)$.
14. $(3x-4)(x+7)$.
15. $(5x-3)(x-1)$.
16. $(5x-3)(x+1)$.
17. $(2x+3)(2x+1)$.
18. $(2x-3)(2x-1)$.
19. $(2x-3)(2x+1)$.
20. $(4x-3)(x-1)$.
21. $(3x-1)(2x+5)$.
22. $(6x+1)(x+5)$.
23. $(7x-4)(x-2)$.
24. $(7x-8)(x-1)$.
25. $(2x+3)(x-3)$.
26. $(2x+9)(x-1)$.
27. $(2x+1)(5x+7)$.
28. $(10x-7)(x-1)$.
29. $(3x+5)(x-3)$.
30. $(3x-3)(x+5)$.
31. $(13x+2)(x-1)$.
32. $(12x+3)(x+1)$.
33. $(2x+3)(6x-1)$.

**34.** $(12x+1)(x-3)$. **35.** $(5x-3)(x+5)$. **36.** $(5x-3)(x-5)$.
**37.** $(5x-1)(x+15)$. **38.** $(4x+7)(2x-1)$. **39.** $(8x+7)(x-1)$.
**40.** $(2x-3)(x+8)$.

### Pages 361-363

**1.** 3 or 2. **2.** $-3$ or $-4$. **3.** 4 or $-3$. **4.** 5 or 2.
**5.** 7 or 4. **6.** $-10$ or $-5$. **7.** 1 or 2. **8.** 8 or $-7$.
**9.** 0 or 7. **10.** 0 or $-5$. **11.** 6 or $-6$. **12.** 7 or $-7$.
**13.** $-3$ or $-\frac{1}{2}$. **14.** $-5$ or $-\frac{2}{3}$. **15.** $\frac{1}{2}$ or $-3$. **16.** 2 or $-2$.
**17.** 0 or 4. **18.** $-2$ or $-\frac{2}{5}$. **19.** $\frac{1}{3}$ or $-\frac{3}{4}$. **20.** 5 or $\frac{5}{3}$.
**21.** 25 or 3. **22.** $\frac{3}{2}$ or $-3$. **23.** $\frac{7}{8}$ or $-1$. **24.** 25 or $-4$.
**25.** 8, 9 or $-9$, $-8$. **26.** $-8$, $-7$. **27.** 8, 10 or $-10$, $-8$.
**28.** 30 rods by 20 rods. **29.** 9 in. by 13 in. **30.** 6, 7 or $-7$, $-6$.
**31.** $\frac{3}{2}$ or $-3$. **32.** 3 in., 4 in., and 5 in.
**33.** 12 in. by 8 in. **34.** 10 in. by 13 in.
**35.** 25 and 81. **36.** 10 in. by 15 in.

### Page 366

**34.** (a) $x^2-4y^2$, (b) $x^3+y^3$, (c) $15a^2-16ab-15b^2$.
**35.** (a) $16x^2-8xy+y^2$, (b) $25x^2+40xy+16y^2$,
(c) $(x^2+2xy+y^2)/4z^2$.
**36.** $x^3-6x^2y+12xy^2-8y^3$. **37.** $\frac{1}{2}(2a^2+5ab-3b^2)$.
**38.** $6x^2+4xy$ cu. in. **39.** $\pi(3a-b)^2(3a+b)$ cu. ft.
**40.** $\pi(3m+n)(5m-n)$ sq. in.

### Page 368

**1.** $2x^2-2xy+4y^2$. **2.** $4bc+2ca-b^2$.
**3.** $6x-2$. **4.** $6a^2b+2b^3$.
**5.** $-7x^2-36xy+10y^2$. **6.** $4m^2+48mn+11n^2$.
**7.** $-22x^2+37x+1$. **8.** $15a^4-a^3b-19a^2b^2+9ab^3-4b^4$.
**9.** $3x^2-6xy-6xz$. **10.** $4k^2+27km-16m^2$.
**11.** $2x^3-32x^2+46x+4$. **12.** $8x^2+50x+470$.
**13.** $16x^2+20x-60$. **14.** $-41x^3+30x^2-51x+74$.
**15.** $6a^2+4b^2-2c^2$. **16.** $4y^2$.

17. $16p^3-24p^2q+12pq^2-2q^3$.
18. $9x^2-17x+8$.
19. $4km-8mn+4kn$.
20. $330x-3067$.

**Pages 370-371**

1. $-2x^2+xy-3y^2$.
2. $2a+3b-a^2$.
3. $x-2$.
4. $a-5b$.
5. $x^2+1$.
6. $4x^2+2xy+y^2$.
7. $3x^2-2x+1$.
8. $x+4y$.
9. $x^2+x-4$.
10. $a^2+2a+5$.
11. $7n^2+8n-5$.
12. $x^2-2xy-3y^2$.
13. $a^2+a+1$.
14. $4x^2-x+3$.
15. $a+b$.
16. $-m^2-3m-1$.
17. $-2a^2+a+2$.
18. $z^2+3z+2$.
19. $x^2+xy+y^2$.
20. $a^2-ab+b^2$.
21. $3a-5$.
22. $m^2-mn+n^2$.
23. $2(x^2+xy+y^2)$.

**Pages 372-373**

1. $4(x-2y)$.
2. $a(x^2+y^2)$.
3. $(x-3)(x-5)$.
4. $(m-n)(m+n)$.
5. $3x(x-2)$.
6. $(a-3)(a+3)$.
7. $(2x-1)(x+1)$.
8. $2(x-2)(x+2)$.
9. $5xy(2xy+3)$.
10. $(2x-1)(2x+1)$.
11. $(1-3m)(1+3m)$.
12. $(x-8)(x+5)$.
13. $z(z-1)(z+1)$.
14. $2b(b^2-2b+3)$.
15. $a(x+3)(x+2)$.
16. $x^2(x-1)(x+1)$.
17. $3y^2(y-2)(y+2)$.
18. $mx^2(x-1)$.
19. $5(a-2)(a+2)$.
20. $4(x-3)(x-5)$.
21. $12a(3b-a)$.
22. $2x(x-3)(x+1)$.
23. $(3m+1)(m-1)$.
24. $(x-\frac{1}{2})(x+\frac{1}{2})$.
25. $(n-4)(n-9)$.
26. $(3xy-1)(3xy+1)$.
27. $5(2x-1)(x+2)$.
28. $(1-k)(1+k)$.
29. $(1-2km)(1+2km)$.
30. $ab^2(b+1)$.
31. $6xy(x^2+2xy-3)$.
32. $(x-1)(x+1)(x^2+1)$.
33. $(1-a)(1+a)(1+a^2)$.
34. $xy(x+1)^2$.
35. $2(a-13)(a+2)$.
36. $(2x-1)(3x+4)$.

37. $k(m-5)(m+5)$.
38. $3(2x-5)(x+2)$.
39. $(a-b)(a+b)(a^2+b^2)$.
40. $(a+b-c)(a+b+c)$.
41. $(x-y-z)(x-y+z)$.
42. $(2c+d-2a)(2c+d+2a)$.
43. (a) $(x-5)^2$; (b) $(x-5)$ in.
44. (a) $(m+9)$ ft., (b) $(2y-1)$ ft. (c) $(b-6)$ ft.

**Page 375**

1. $(m^2+n^2+mn)(m^2+n^2-mn)$.
2. $(a-2)(a+2)(a-1)(a+1)$.
3. $(x^2+2+x)(x^2+2-x)$.
4. $(y-3)(y-1)(y+3)(y+1)$.
5. $(2m^2+2m+1)(2m^2-2m+1)$.
6. $(2c^2+c+1)(2c^2-c+1)$.
7. $(a^2-b^2+4ab)(a^2-b^2-4ab)$.
8. $(a^2+b^2+4ab)(a^2+b^2-4ab)$.
9. $25(2p^2+1+2p)(2p^2+1-2p)$.
10. $(k^2-1+k)(k^2-1-k)$.
11. $(4x^2+5y^2+2xy)(4x^2+5y^2-2xy)$.
12. $(m^2+m+1)(m^2-m+1)(m^4-m^2+1)$.
13. $(5x+8y)(x-y)(5x-8y)(x+y)$.
14. $(5x^2+8y^2+6xy)(5x^2+8y^2-6xy)$.

**Page 377**

4. $(1-4c)(1+4c+16c^2)$.
5. $(3x+1)(9x^2-3x+1)$.
6. $(p+2q)(p^2-2pq+4q^2)$.
7. $(2a-b)(4a^2+2ab+b^2)$.
8. $(x+4y)(x^2-4xy+16y^2)$.
9. $(3m-2)(9m^2+6m+4)$.
10. $(5m-2n)(25m^2+10mn+4n^2)$.
11. $(1+2cd)(1-2cd+4c^2d^2)$.
12. $(6p-10q)(36p^2+60pq+100q^2)$.
13. $(10ab+1)(100a^2b^2-10ab+1)$.
14. $2(m+n)(m^2-mn+n^2)$.
15. $(10-ab)(100+10ab+a^2b^2)$.
16. $(k^2-m)(k^4+k^2m+m^2)$.
17. $(a^2+b^2)(a^4-a^2b^2+b^4)$.
18. $2(x+3)(x^2-3x+9)$.
19. $3(k-2)(k^2+2k+4)$.
20. $d^2(1+d)(1-d+d^2)$.
21. $3x(x-4y)(x^2+4xy+16y^2)$.

**Pages 379-380**

1. Both are factors.
2. $x-2$ is a factor.
3. Both are factors.
4. $a+2$ is a factor.
5. $a+1$ is a factor.
6. Both are factors.
7. $(x-1)(x-2)(x+1)$.
8. $(x-2)(x^2+3x+1)$.
9. $(x-2)(x-1)(x+2)$.
10. $(a+2)(a^2+1)$.

11. $(a+1)(a+3)(a-1)$.
12. $(a+1)(a+2)(a-1)$.
13. $(x-1)(x^2-3x+1)$.
14. $(x-2)(x-3)(x+1)$.
15. $(x+1)(x^2+x+4)$.
16. $(x+2)(x^2+3x-8)$.

### Page 381

1. $(x-7)(x-5)$.
2. $(x-9)(x+9)$.
3. $(m+1)(m^2-m+1)$.
4. $ab(a-2b)(a+2b)$.
5. $(2x-3)(x+2)$.
6. $(2a-3)(4a^2+6a+9)$.
7. $3a(2b+c-3d)$.
8. $(5a-7)(5a+7)$.
9. $(5a-2)(25a^2+10a+4)$.
10. $(3c+5)(2c-1)$.
11. $(r-1)(r+1)(r^2+1)$.
12. $2x^2y^2(2x-y)$.
13. $3(5k-3)(5k+3)$.
14. $8(n+2)(n^2-2n+4)$.
15. $(2a-5)(a-4)$.
16. $3x(2x+3)(4x^2-6x+9)$.
17. $(s^2+1+s)(s^2+1-s)$.
18. $3a^2b(2ab-1)(ab-1)$.
19. $5(2x-1)(4x^2+2x+1)$.
20. $(p-2)(p+2)(p-1)(p+1)$.
21. $(x-2y)(x+2y)(x-y)(x+y)$.
22. $6(n-3)(n^2+3n+9)$.
23. $k(r+1)(r^2-r+1)$.
24. $(ab-\frac{1}{2})(ab+\frac{1}{2})$.
25. $(3x-5)(x+7)$.
26. $7ax(x+2)^2$.
27. $(t-2)^2(t+2)^2$.
28. $y(y-3)^2$.
29. $(1-xy^2)(1+xy^2)$.
30. $(7a-3)(3a-2)$.
31. $m^2n(m-4n)(m-n)$.
32. $r^2(r-2)(r+2)$.
33. $(1-a)(1+a+a^2)(1+a)(1-a+a^2)$.
34. $(k^2-1+k)(k^2-1-k)$.
35. $5xy^2(xy+2)(x^2y^2-2xy+4)$.
36. $(6x-7)(2x+1)$.

### Pages 384-386

1. $\frac{3}{4}$.
2. $x/a$.
3. $\frac{1}{3}$.
4. $xz/a$.
5. $\frac{22}{15}$.
6. $3c/b$.
7. $m/n$.
8. $\frac{3}{2}$.
9. $\dfrac{x-3}{x+3}$.
10. $(a-b)$.
11. $\dfrac{2}{x+2}$.
12. $\dfrac{m-4}{m+3}$.
13. $2xy$.
14. $\dfrac{2(x-5)}{x+2}$.
15. $2(a-5)$.

16. $\frac{1}{m^2+2m+4}$.
17. $\frac{2x+3}{4x^2+6x+9}$.
18. $\frac{c+d}{c^2+d^2+cd}$.
19. $\frac{6}{7}$.
20. $\frac{3}{2}$.
21. $\frac{6}{5}$.
22. $x$.
23. $mn/3$.
24. $3a/2b$.
25. $\frac{3}{5}$.
26. $x(x-y)$.
27. 5.
28. $(a-4)$.
29. 2.
30. 1.
31. $\frac{x+7}{4}$.
32. $mn$.
33. $\frac{x^2-xy+y^2}{x^2+xy+y^2}$.
34. $\frac{a-3}{4a^2+2a+1}$.
35. $\frac{1}{m+1}$.
36. 1.
37. $\frac{a-2}{a-6}$.
38. $m/nx$.
39. 1.
40. 1.
41. 1.
42. $\frac{(x-1)(x^2+x+1)}{(y-1)(y+1)}$.
43. $\frac{125(k+m)}{27}$.
44. $(x+7)$.
45. $\frac{3x-1}{x}$.
46. $\frac{a^2+b^2}{a+b}$.
47. $\frac{2m^2+1-2m}{4m^2+2m+1}$.
48. $c^2+2c+4$.

**Pages 391-392**

1. 12.
2. 72.
3. $abcx$.
4. $18y^3$.
5. $km^2n^2$.
6. $b^2x^3$.
7. $x^2-y^2$.
8. $(x-y)^2$.
9. $(a+b)(c+d)$.
10. $(a-5)(a-2)(a-3)$.
11. $(x+y)^2(x-y)$.
12. $bx^2(x-y)$.
13. $(a^2-b^2)(a^2+ab+b^2)$.
14. $(m+2)^2(m^2-2m+4)$.
15. $\frac{5}{6}$.
16. $\frac{y+x}{xy}$.
17. $\frac{1}{3}$.
18. $\frac{3c-5}{bc}$.
19. $\frac{13}{10}$.
20. $\frac{5z+7x}{xyz}$.
21. $\frac{13}{14}$.
22. $\frac{k(2n-5)+7m}{kmn}$.
23. $\frac{91}{180}$.
24. $\frac{13az-7ay+11yz}{xyza}$.
25. $\frac{9a+b}{20}$.
26. $\frac{5(2x-y)}{24}$.

27. $\frac{x-5y}{(x+y)(x-y)}$.

28. $\frac{xy}{(x^2-y^2)}$.

29. $\frac{ab}{(a+b)^2}$.

30. $\frac{2x}{(x+y)^2(x-y)}$.

31. $\frac{-4xy}{(x-y)(x+y)}$.

32. $\frac{a^2+b^2}{a(a-b)}$.

33. $\frac{x^2+xy+y^2+x}{x^3-y^3}$.

34. $\frac{a^2-ab+b^2-b}{a^3+b^3}$.

35. 0.

36. $\frac{2(x^2-5x+5)}{(x-2)(x-4)}$.

37. $\frac{2x}{(x-1)(x+1)}$.

38. $\frac{mn}{(m+n)(m-n)}$.

39. $\frac{2(x-6y)}{x(x+y)(x-y)}$.

40. $\frac{2a+5}{(a+1)(a+2)(a+3)}$.

41. 0.

42. 0.

43. $\frac{(b-a)(2a-b)}{ab}$.

44. 0.

45. $\frac{1-a}{(a-3)(a-5)}$.

46. $\frac{15}{(x+5)(x-5)}$.

47. $\frac{a^2+b^2-ab-1}{(a^2+b^2+ab)(a^2+b^2-ab)}$.

48. $\frac{6x+23}{8x^3-27}$.

**Pages 394-395**

4. 0.

5. $\frac{10}{3x-y}$.

6. $\frac{c}{c^2-4d^2}$.

7. $\frac{mn}{m^2-n^2}$.

8. 0.

9. 0.

10. $\frac{2a}{3a-2b}$.

11. $\frac{10}{(k-1)(k+1)}$.

**Pages 396-398**

1. 24.

2. 4.

3. $\frac{10}{3}$.

4. $-6$.

5. 55.

6. $-1$.

7. 2. 8. 4. 9. 1.
10. 15. 11. 1. 12. $-\frac{1}{2}$.
13. 6. 14. $-\frac{1}{3}$. 15. 1.
16. 7. 17. $-7$. 18. $\frac{3}{7}$.
19. $-\frac{4}{5}$. 20. $-2$. 21. $-\frac{1}{2}$ and 3.
22. $\pm 2$. 23. 60. 24. 20 and 21.
25. 24. 26. $\frac{2}{5}$. 27. $\frac{5}{3}$.
28. 1. 29. $\frac{4}{3}$ or $\frac{1}{2}$. 30. $\frac{11}{6}$.
31. $-\frac{5}{4}$. 32. 4 or $-1$. 33. 280/3 and 20/3.
34. $-13$. 35. 6 or $-\frac{1}{2}$.
36. $100/x$ m.p.h.; $100/x$ hrs.; $5x$ miles; $40x$ miles; $x/30$ hrs.; $x/10$ m.p.h.
37. 40 m.p.h. 38. 2 m.p.h.

**Pages 399-400**

1. $2x^2(x-2y)$. 2. $(m+6)(m+1)$. 3. $(x-y)(x+y)$.
4. $(2x-3)(x+2)$. 5. $2(n-2)(n+2)$. 6. $3ab(3a-2b+1)$.
7. $(x-4)(x^2+4x+16)$. 8. $(3m+1)(9m^2-3m+1)$.
9. $(x-3)(x+3)(x-1)(x+1)$. 10. $a^2+4ab+4b^2$.
11. $4x^2-4xy+y^2$. 12. $a^2+b^2+4c^2+2ab+4ac+4bc$.
13. $m^3+3m^2n+3mn^2+n^3$. 14. $r^3-9r^2+27r-27$.
15. $x^3+9x^2y+27xy^2+27y^3$. 16. $2(3n^2-mn)$.
17. $x^2+y^2-z^2+4xy$. 18. $2(x^3-6x^2y+12xy^2-8y^3)$.
19. $-a(45a+44b)$. 20. $2m^4-m^3n-3m^2n^2+10mn^3-8n^4$.
21. $9a^2+9a+5$. 22. $x-2$. $4x^2/5y$.
24. $\dfrac{(x-y)}{2(x+y)}$. 25. $\frac{4}{21}$. 26. $\dfrac{5}{x^2-y^2}$.
27. $x$. 28. $12(x^2-y^2)(x-y)$. 29. $\dfrac{1}{x-y}$. 30. 0.

**Pages 400-401**

1. $x^2(x-3)(x+3)$. 2. $(3x-5y)^2$.
3. $3x^2(2x-y)$. 4. $3(a+3)(a^2-3a+9)$.
5. $m(m-2)(m^2+2m+4)$. 6. $(t^2+1+t)(t^2+1-t)$.
7. (i) $2(a^2+b^2)$; (ii) $4ab$. 8. $72y^2-38xy$.

9. $7+2a-20a^2$.
10. $x^2+y^2+z^2-6xy-14xz-2yz$.
11. $x^2+3x+3$.
12. $2x-3$.
13. $\dfrac{3x+4}{x-2}$.
14. $\dfrac{m-5}{m-1}$.
15. 0.
16. $\dfrac{-2}{(a+5)(a-2)(a+3)}$.
17. $x=5$.
18. 50 and 51.
19. $x=-\frac{3}{5}$ or 4.
20. 14.

**Pages 401-402**

1. $(1-2x)(1+2x)$.
2. $b(1-3a)(1+3a)$.
3. $3(2x-1)(x+1)$.
4. $ab(a-4b)(a+4b)$.
5. $5m^2n^2(2m-1)(m+2)$.
6. $(k-\frac{1}{3})(k+\frac{1}{3})$.
7. $3xy(x+2y)(x^2-2xy+4y^2)$.
8. $(x-1)^3$.
9. $(x-2y-5z)(x-2y+5z)$.
10. $54k^3+72kn^2$.
11. $56x^2$.
12. $2x^2-5x+3$.
13. $4a^2-2a+7$.
14. $10x^4+x^3y-6x^2y^2+7xy^3-4y^4$.
15. $\frac{9}{10}$.
16. $a/bc$.
17. $\dfrac{3m}{m+6}$.
18. $\dfrac{x+3}{x+5}$.
19. $\frac{1}{6}(4x-3y)$.
20. $\dfrac{n-m+1}{m+n}$.
21. $x=3$.
22. $a=2$.
23. 240 and 120.

**Pages 402-404**

1. $(4z+1)(z-4)$.
2. $(x+y-z)(x+y+z)$.
3. $(5a-2b)(25a^2+10ab+4b^2)$.
4. $(2m+1)(2m-1)(m+1)(m-1)$.
5. $5(2c-3d)(4c^2+6cd+9d^2)$.
6. $(x+1)(x^2+1)$.
7. $z=4$.
8. $4, \frac{1}{2}$.
9. $4a^2$.
10. $25m^2+48n^2$.
11. 0.
12. $\frac{5}{2}\frac{5}{4}$.
13. $axy/b$.
14. $x$.
15. 0.
16. $4/m^2n^2$.

17. $\frac{c+d}{r+s}$.
18. 0.
19. $\frac{14(x-1)}{(x-5)(x+2)}$.
20. $\frac{2}{k^2-8k+15}$.
21. 15.
22. $\frac{1}{4}$.
23. 6 m.p.h.

**Pages 404-405**

1. $(5x-2y)(5x+2y)$.
2. $(ab-c)(ab+c)$.
3. $(xy-z)(x^2y^2+xyz+z^2)$.
4. $(6x-5)(2x+3)$.
5. $(1+10rst)(1-10rst+100\,r^2s^2t^2)$.
6. $(a^2+3b^2-2ab)(a^2+3b^2+2ab)$.
7. $(a-2)(a-3)(a+1)$.
8. $(8x+9)(3x+2)$.
9. $(x+2y-3z)(x+2y+3z)$.
10. $2b(b-22a)$.
11. $-19x^3+85x^2-40x$.
12. $-6x^4-5x^3y+10x^2y^2+5xy^3-4y^4$.
13. $a=2$.
14. $4a^2+2ab+b^2$.
15. 5.
16. $\frac{a-10}{2(a+2)(a-2)}$.
17. 1.
18. $\frac{-4y}{3(x-y)(x+y)}$.
19. $\frac{c^2+4}{(c-1)(c+1)}$.
20. $\frac{m}{(m-3n)}$.
21. $x=11$.
22. $x=1$.
23. $x=-\frac{1}{2}$.
24. $\frac{1}{2}\frac{8}{7}$.
25. $x=6$ or $-8$.

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